The Representation Category of the Quantum Group of a Non-Degenerate Bilinear Form

Julien Bichon

Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour, IPRA, Avenue de l'université, 64000 Pau, France. E-mail: Julien.Bichon@univ-pau.fr

Abstract

We show that the representation category of the quantum group of a non-degenerate bilinear form is monoidally equivalent to the representation category of the quantum group $SL_q(2)$ for a well chosen non-zero parameter q. The key ingredient for the proof of this result is the direct and explicit construction of an appropriate Hopf bigalois extension. Then we get, when the base field is of characteristic zero, a full description of cosemisimple Hopf algebras whose representation semi-ring is isomorphic to the one of $SL(2)$.

Keywords: Hopf algebra, monoidal category, Hopf-Galois extension.

1 Introduction and main results

Let k be a commutative algebraically closed field, let $n \in \mathbb{N}^*$, $n \geq 2$ and let $E \in GL(n)$. We consider the following algebra $\mathcal{B}(E)$: it is the universal algebra with generators $(a_{ij})_{1\leq i,j\leq n}$ satisfying the relations

$$E^{-1}{}^t aEa = I = aE^{-1}{}^t aE,$$

where a is the matrix $(a_{ij})_{1\leq i,j\leq n}$ and I is the identity matrix. This algebra admits a natural Hopf algebra structure and was introduced by M. Dubois-Violette and G. Launer [5]. It is the function algebra on the quantum (symmetry) group of a non-degenerate bilinear form (see section 2). Let $q \in k^*$. For a well chosen matrix $E_q \in GL(2)$, we have $\mathcal{B}(E_q) = \mathcal{O}(SL_q(2))$, the function algebra on the quantum group $SL_q(2)$. The main result of this paper describes the category of comodules over $\mathcal{B}(E)$ for a general matrix E:

Theorem 1.1 *Let $E \in GL(n)$, $n \geq 2$, and let $q \in k^*$ be such that $q^2 + \mathrm{tr}(E^t E^{-1})q + 1 = 0$. Then we have an equivalence of monoidal categories:*

$$\mathrm{Comod}(\mathcal{B}(E)) \cong^{\otimes} \mathrm{Comod}(\mathcal{O}(SL_q(2)))$$

between the comodule categories of $\mathcal{B}(E)$ and $\mathcal{O}(SL_q(2))$ respectively.

When $k = \mathbb{C}$, T. Banica [1] proved a related result, describing the representation semi-ring of the compact analogues of $\mathcal{B}(E)$ introduced by A. Van Daele and S.Wang [17]. Theorem 1.1 covers the cosemisimple non-compact case as well as the non-cosemisimple case. There are also other related results in the literature, in the $SL(N)$ case: again by Banica [2] in the compact case and by Phung Ho Hai [7] in the cosemisimple case (q is not a root of unity) in characteristic zero. In these two approaches the authors study Hopf algebras reconstructed from Hecke symmetries. When $N = 2$, we do not even have to mention Hecke symmetries: indeed by D. Gurevich's classification of even Hecke symmetries of rank two [6], all such Hecke symmetries may be constructed using bilinear forms.

We wish to emphasize that our result is characteristic-free and does not depend on the cosemisimplicity of the considered Hopf algebras. The main reason is that our technique of proof is different from the one of Banica and Phung Ho Hai. These two authors use reconstruction techniques. Here we directly construct an explicit $\mathcal{O}(SL_q(2))$-$\mathcal{B}(E)$-bigalois extension: by a very useful theorem of P. Schauenburg [15] (see also K.H. Ulbrich [16]), this is equivalent to constructing an equivalence of monoidal categories between the comodule categories of these two Hopf algebras. The technical difficulty in our approach is to show that the algebra we construct is non-zero. Since the the monoidal equivalences we get do not preserve the dimensions of the underlying vector spaces in general, our Galois extensions will be non-cleft in general. The existence of non-cleft Hopf-Galois extensions was known: first by the end of the paper [4] of A. Bruguières, and also by the results of Banica and Phung Ho Hai. However it is the first time, at least to the best of our knowledge, that non-cleft Hopf-Galois extensions are explicitly described.

Let us point out a negative consequence of Theorem 1.1 in the perspective of knot theory. Recall that the Jones polynomial may be constructed from the representation category of the quantum group $SL_q(2)$ (see the book [9]). Theorem 1.1 means that one cannot expect to get any new link invariant from the more general Hopf algebras $\mathcal{B}(E)$.

We also prove a kind of converse to Theorem 1.1: the description of all cosemisimple Hopf algebras whose representation semi-ring is isomorphic to the one of $SL(2)$. Here we have to assume that the characteristic of k is zero. We say that an element $q \in k^*$ is generic if $q \in \{\pm 1\}$ or if q is not a root of unity.

Theorem 1.2 *Let A be a cosemisimple Hopf algebra whose representation semi-ring is isomorphic to the one of $SL(2)$. Then there exists $E \in GL(n)$ ($n \geq 2$) such that A is isomorphic with $\mathcal{B}(E)$, and such that any solution of the equation $q^2 + \operatorname{tr}(E^tE^{-1})q + 1 = 0$ is generic. If $F \in GL(m)$ is another matrix such that A is isomorphic with $\mathcal{B}(F)$, then $n = m$ and there exists $M \in GL(m)$ such that $F = {}^tMEM$*

Once again an analogue of Theorem 1.2 was proved in the compact quantum group case in [1], namely the description of all compact quantum groups having a representation semi-ring isomorphic to the one of $SU(2)$. But again we have here the cosemisimple non-compact case. Theorem 1.2 was already known if one requires the fundamental comodule of A to be of dimension 2, partially by results of S.L. Woronowicz [20], a complete proof being given in P. Podleś and E. Müller's notes [14]. The $SL(3)$-case has been done by C.

Ohn [13] with a constraint on the dimension of the fundamental comodule. Finally the compact case $SU(N)$ was done in [2], without any dimension constraint but without an isomorphic classification.

Theorem 1.1 is used in an essential way to prove Theorem 1.2. The other main ingredient for the proof of Theorem 1.2 is the representation theory of $SL_q(2)$, including the root of unity case (see [11]). The strategy of proof is then the same as the one of Podleś and Müller [14].

Our paper is organized as follows. In Section 2 we briefly recall some facts concerning the Hopf algebras $\mathcal{B}(E)$. In Section 3, we associate an algebra $\mathcal{B}(E,F)$ to each pair (E,F) of matrices. It is shown that if $\mathcal{B}(E,F)$ is a non zero-algebra, then $\mathcal{B}(E,F)$ is a $\mathcal{B}(E)$-$\mathcal{B}(F)$-bigalois extension. In section 4 we prove, using the diamond Lemma [3], that $\mathcal{B}(E_q,F)$ is a non-zero algebra for a well chosen $q \in k^*$: this proves Theorem 1.1. In section 5 we prove Theorem 1.2 and describe the isomorphic classification of the Hopf algebras $\mathcal{B}(E)$ (in characteristic zero). Finally we study possible CQG algebra structures on $\mathcal{B}(E)$ in Section 6.

Throughout this paper k is an algebraically closed field.

2 The Hopf algebras $\mathcal{B}(E)$

In this section we briefly recollect some basic results (without proofs) concerning the Hopf algebras $\mathcal{B}(E)$.

Let $n \in \mathbb{N}^*$ and let $E \in GL(n)$. We have already defined the algebra $\mathcal{B}(E)$. It was introduced by M. Dubois-Violette and G. Launer in [5]. The following result is taken from [5]:

Proposition 2.1 *The algebra $\mathcal{B}(E)$ admits a Hopf algebra structure, with comultiplication Δ defined by $\Delta(a_{ij}) = \sum_{k=1}^n a_{ik} \otimes a_{kj}$, $1 \leq i,j \leq n$, with counit ε defined by $\varepsilon(a_{ij}) = \delta_{ij}$, $1 \leq i,j \leq n$, and with antipode S defined on the matrix $a = (a_{ij})$ by $S(a) = E^{-1}\,{}^t aE$.* $\square$

The Hopf algebra $\mathcal{B}(E)$ was defined in [5] as the function algebra on the quantum group of a bilinear form associated with E. This is explained by the following result, which was not explicitly stated in [5], but was clearly implicit in that paper:

Proposition 2.2 *i) Consider the vector space $V = k^n$ with its canonical basis $(e_i)_{1 \leq i \leq n}$. Endow V with the $\mathcal{B}(E)$-comodule structure defined by $\alpha(e_i) = \sum_{j=1}^n e_j \otimes a_{ji}$, $1 \leq i \leq n$. Then the linear map $\beta : V \otimes V \longrightarrow k$ defined by $\beta(e_i \otimes e_j) = \lambda_{ij}$, $1 \leq i,j \leq n$, where $E = (\lambda_{ij})$, is a $\mathcal{B}(E)$-comodule morphism.*
ii) Let A be a Hopf algebra and let V be a finite-dimensional A-comodule of dimension n. Let $\beta : V \otimes V \longrightarrow k$ be an A-comodule morphism such that the associate bilinear form is non-degenerate. Then there exists $E \in GL(n)$ such that V is a $\mathcal{B}(E)$-comodule, such that β is a $\mathcal{B}(E)$-comodule morphism, and there exists a unique Hopf algebra morphism $\phi : \mathcal{B}(E) \longrightarrow A$ such that $(\mathrm{id}_V \otimes \phi) \circ \alpha = \alpha'$, where α and α' denote the coactions on V of $\mathcal{B}(E)$ and A respectively. $\square$

The next result was also known in [5]. It will be generalized at the Hopf-Galois extension level in the next section.

Proposition 2.3 *Let $E, P \in GL(n)$. Then the Hopf algebras $\mathcal{B}(E)$ and $\mathcal{B}({}^tPEP)$ are isomorphic.* $\square$

We end the section by connecting the Hopf algebras $\mathcal{B}(E)$ with the Hopf algebra $\mathcal{O}(SL_q(2))$. Let $q \in k^*$ and let $E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \in GL(2)$. Then it is a straightforward computation to check that $\mathcal{B}(E_q) = \mathcal{O}(SL_q(2))$ (with the definition of [9] for $\mathcal{O}(SL_q(2))$).

3 The Hopf bigalois extensions $\mathcal{B}(E, F)$

In order to prove Theorem 1.1, we introduce appropriate Hopf bigalois extensions. By Schauenburg's Theorem 5.5 in [15], it is equivalent to construct Hopf bigalois extensions and monoidal equivalences between comodule categories. Let us first recall the language of Galois extensions for Hopf algebras (see [12] for a general perspective).

Let A be a Hopf algebra. A left A-Galois extension (of k) is a non-zero left A-comodule algebra Z such that the linear map κ_l defined by the composition

$$\kappa_l : Z \otimes Z \xrightarrow{\alpha \otimes 1_Z} A \otimes Z \otimes Z \xrightarrow{1_A \otimes m_Z} A \otimes Z$$

where α is the coaction of A and m_Z is the multiplication of Z, is bijective.

Similarly, a right A-Galois extension is a non-zero right A-comodule algebra Z such that the linear map κ_r defined by the composition

$$\kappa_r : Z \otimes Z \xrightarrow{1_Z \otimes \beta} Z \otimes Z \otimes A \xrightarrow{m_Z \otimes 1_A} Z \otimes A$$

where β is the coaction of A, is bijective.

Let A and B be Hopf algebras. An algebra Z is said to be an A-B-bigalois extension [15] if Z is both a left A-Galois extension and a right B-Galois extension, and if Z is an A-B-bicomodule. By Theorem 5.5 in [15], there exists a monoidal equivalence between the categories Comod(A) and Comod(B) if and only if there exists an A-B-bigalois extension.

The following definition is a natural generalization of the definition of the algebras $\mathcal{B}(E)$:

Definition 3.1 *Let $E \in GL(m)$ and let $F \in GL(n)$. The algebra $\mathcal{B}(E, F)$ is the universal algebra with generators z_{ij}, $1 \le i \le m, 1 \le j \le n$, satisfying the relations*

$$F^{-1}{}^tzEz = I_n \ ; \ zF^{-1}{}^tzE = I_m.$$

We have $\mathcal{B}(E, E) = \mathcal{B}(E)$. Let us first prove a generalization of Proposition 2.3:

Proposition 3.2 *Let $E, P \in GL(m)$ and let $F, Q \in GL(n)$. Then the algebras $\mathcal{B}(E, F)$ and $\mathcal{B}({}^tPEP, {}^tQFQ)$ are isomorphic.*

Proof. Let us denote by y_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, the generators of $\mathcal{B}({}^tPEP, {}^tQFQ)$. Then the relations

$$(Q^{-1}F^{-1t}Q^{-1})^t y({}^tPEP)y = I_n \quad \text{and} \quad y(Q^{-1}F^{-1t}Q^{-1})^t y({}^tPEP) = I_m$$

ensure that we have an algebra morphism $\psi : \mathcal{B}(E, F) \longrightarrow \mathcal{B}({}^tPEP, {}^tQFQ)$ defined by $\psi(z) = PyQ^{-1}$. The inverse map is then defined by $\psi^{-1}(y) = P^{-1}zQ$. $\square$

The algebras $\mathcal{B}(E, F)$ are natural candidates to be $\mathcal{B}(E)$-$\mathcal{B}(F)$-bigalois extensions. We now define several structural maps. Let us first fix some notations. The generators of $\mathcal{B}(E)$ are denoted by a_{ij}, $1 \leq i, j \leq m$; the generators of $\mathcal{B}(F)$ are denoted by b_{ij}, $1 \leq i, j \leq n$; the generators of $\mathcal{B}(E, F)$ are denoted by z_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$; the generators of $\mathcal{B}(F, E)$ are denoted by y_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$.

The reader will easily check that the algebra morphisms described below are well-defined, and that they are coassociative.

• The algebra morphism $\alpha : \mathcal{B}(E, F) \longrightarrow \mathcal{B}(E) \otimes \mathcal{B}(E, F)$ defined by

$$\alpha(z_{ij}) = \sum_{k=1}^{m} a_{ik} \otimes z_{kj}, \ 1 \leq i \leq m, 1 \leq j \leq n,$$

endows $\mathcal{B}(E, F)$ with a left $\mathcal{B}(E)$-comodule algebra structure.

• Similarly, the algebra morphism $\beta : \mathcal{B}(E, F) \longrightarrow \mathcal{B}(E, F) \otimes \mathcal{B}(F)$ defined by

$$\beta(z_{ij}) = \sum_{k=1}^{n} z_{ik} \otimes b_{kj}, \ 1 \leq i \leq m, 1 \leq j \leq n,$$

endows $\mathcal{B}(E, F)$ with a right $\mathcal{B}(F)$-comodule algebra structure. It is clear that $\mathcal{B}(E, F)$ is a $\mathcal{B}(E)$-$\mathcal{B}(F)$-bicomodule.

We need several other algebra morphisms to prove that the maps κ_l and κ_r are bijective. Once again it is straightforward to check that the algebra morphisms considered below are well-defined.

• We have an algebra morphism $\phi : \mathcal{B}(F, E) \longrightarrow \mathcal{B}(E, F)^{\mathrm{op}}$ defined by

$$\phi(y) = F^{-1t}zE.$$

• We have an algebra morphism $\gamma_1 : \mathcal{B}(E) \longrightarrow \mathcal{B}(E, F) \otimes \mathcal{B}(F, E)$ defined by

$$\gamma_1(a_{ij}) = \sum_{k=1}^{n} z_{ik} \otimes y_{kj}, \ 1 \leq i, j \leq m.$$

Similarly we have an algebra morphism $\gamma_2 : B(F) \longrightarrow \mathcal{B}(F, E) \otimes \mathcal{B}(E, F)$ defined by

$$\gamma_2(b_{ij}) = \sum_{k=1}^{m} y_{ik} \otimes z_{kj}, \ 1 \leq i, j \leq n.$$

We have introduced all the ingredients to prove the following result:

Proposition 3.3 *Assume that $\mathcal{B}(E,F) \neq 0$. Then $\mathcal{B}(E,F)$ is a $\mathcal{B}(E)$-$\mathcal{B}(F)$-bigalois extension.*

Proof. Let η_l be the unique linear map such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{B}(E) \otimes \mathcal{B}(E,F) & \xrightarrow{\eta_l} & \mathcal{B}(E,F) \otimes \mathcal{B}(E,F) \\ {\scriptstyle \gamma_1 \otimes 1} \downarrow & & \uparrow {\scriptstyle 1 \otimes m} \\ \mathcal{B}(E,F) \otimes \mathcal{B}(F,E) \otimes \mathcal{B}(E,F) & \xrightarrow{1 \otimes \phi \otimes 1} & \mathcal{B}(E,F) \otimes \mathcal{B}(E,F) \otimes \mathcal{B}(E,F) \end{array}$$

and similarly, let η_r be the unique linear map such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{B}(E,F) \otimes \mathcal{B}(F) & \xrightarrow{\eta_r} & \mathcal{B}(E,F) \otimes \mathcal{B}(E,F) \\ {\scriptstyle 1 \otimes \gamma_2} \downarrow & & \uparrow {\scriptstyle m \otimes 1} \\ \mathcal{B}(E,F) \otimes \mathcal{B}(F,E) \otimes \mathcal{B}(E,F) & \xrightarrow{1 \otimes \phi \otimes 1} & \mathcal{B}(E,F) \otimes \mathcal{B}(E,F) \otimes \mathcal{B}(E,F) \end{array}$$

Now let us note the following identities:

$$\sum_{k=1}^{m} \phi(y_{lk}) z_{kj} = \delta_{lj}, \quad 1 \leq l,j \leq n, \quad \text{and} \quad \sum_{k=1}^{n} z_{lk} \phi(y_{kj}) = \delta_{lj}, \quad 1 \leq l,j \leq m.$$

Let $x \in \mathcal{B}(E,F)$, let $i,k \in \{1,\dots,m\}$ and let $j,l \in \{1,\dots,n\}$. Then using the previous identities, it is immediate to check that

$$\eta_l \circ \kappa_l(z_{ij} \otimes x) = z_{ij} \otimes x \ , \ \kappa_l \circ \eta_l(a_{ik} \otimes x) = a_{ik} \otimes x,$$

and

$$\eta_r \circ \kappa_r(x \otimes z_{ij}) = x \otimes z_{ij} \ , \ \kappa_r \circ \eta_r(x \otimes b_{jl}) = x \otimes b_{jl}.$$

Now using the facts that γ_1, γ_2 and ϕ are algebra morphisms and that the elements considered in these equations are generators of the corresponding algebras, it is not hard to see that η_l and η_r are inverse isomorphisms of κ_l and κ_r respectively. $\square$

We have now to determine when the algebra $\mathcal{B}(E,F)$ is a non-zero algebra. This is done in the next Proposition. Let $q \in k^*$: we use the matrix E_q introduced in the previous section.

Proposition 3.4 *Let $q \in k^*$ and let $F \in GL(n)$, $n \geq 2$. Assume that $q^2 + \mathrm{tr}(F^tF^{-1})q + 1 = 0$. Then $\mathcal{B}(E_q,F)$ is a non-zero algebra.*

The proof of Proposition 3.4 requires some work and will be done in the next section. Taking this result for guaranteed, we can prove Theorem 1.1. Indeed by Proposition 3.4, for $q \in k^*$ satisfying $q^2 + \mathrm{tr}(F^tF^{-1})q + 1 = 0$, the algebra $\mathcal{B}(E_q,F)$ is non-zero algebra. Hence by Proposition 3.3 $\mathcal{B}(E_q,F)$ is a $\mathcal{B}(E_q)$-$\mathcal{B}(F)$-bigalois extension. We can use Theorem 5.5 in [15] : we have an equivalence of monoidal categories

$$\mathrm{Comod}(\mathcal{B}(E_q)) \cong^{\otimes} \mathrm{Comod}(\mathcal{B}(F))$$

and since $\mathcal{B}(E_q) = \mathcal{O}(SL_q(2))$, the proof of Theorem 1.1 is complete. Let us note that this monoidal equivalence also induces a monoidal equivalence between the categories of finite-dimensional comodules.

Remark. The algebra $\mathcal{B}(E, F)$ is non-zero when $\mathrm{tr}(E{}^tE^{-1}) = \mathrm{tr}(F{}^tF^{-1})$. We will prove this fact at the end of section 4.

4 Proof of Proposition 3.4

This section is devoted to the proof of Proposition 3.4. Our strategy is the following one. We write a convenient presentation for $\mathcal{B}(E_q, F)$ and use Bergman's diamond lemma [3] to get several linearly independent elements: this will clearly imply that $\mathcal{B}(E_q, F)$ is a non-zero vector space.

Let $F = (\alpha_{ij}) \in GL(n)$ $(n \geq 2)$ with inverse $F^{-1} = (\beta_{ij})$, and let $q \in k^*$ be such that $q^2 + \mathrm{tr}(F{}^tF^{-1})q + 1 = 0$. This equation may be rewritten in the most convenient form:

$$\mathrm{tr}(F{}^tF^{-1}) = \sum_{i,j} \alpha_{ij}\beta_{ij} = -q - q^{-1}.$$

We would like to be able to assume that $\beta_{nn} = 0$. This will avoid some overlap ambiguities in the presentation of $\mathcal{B}(E_q, F)$. The following elementary well-known lemma will be useful for this purpose. We give a proof for the sake of completeness.

Lemma 4.1 *Let $M = (M_{ij}) \in GL(n)$ $(n \geq 2)$. Then there exists $P \in GL(n)$ such that $({}^tPMP)_{nn} = 0$.*

Proof. First assume that $M_{11} = 0$. Let $P = \sum_{i=1}^{n} E_{n-i+1,i} \in GL(n)$, where the E_{ij}'s denote the standard elementary matrices. Then $({}^tPMP)_{nn} = M_{11} = 0$. Now assume that $M_{11} \neq 0$ and $M_{nn} \neq 0$. Let $\lambda \in k^*$ be such that $\lambda^2 M_{nn} + (M_{n1} + M_{1n})\lambda + M_{11} = 0$. Let $P = \sum_{i=1}^{n-1} E_{ii} + \lambda E_{nn} + E_{1n} \in GL(n)$. Then we have $({}^tPMP)_{nn} = 0$. $\square$

Now take $M = F^{-1}$ and pick $P \in GL(n)$ as in Lemma 4.1: $({}^tPF^{-1}P)_{nn} = 0$. Then by Proposition 3.2, the algebras $\mathcal{B}(E_q, F)$ and $\mathcal{B}(E_q, P^{-1}F{}^tP^{-1})$ are isomorphic and we have $(P^{-1}F{}^tP^{-1})^{-1}_{nn} = ({}^tPF^{-1}P)_{nn} = 0$. Thus we may assume without loss of generality that $\beta_{nn} = 0$.

Let us now study in detail the algebra $\mathcal{B}(E_q, F)$: it is the universal algebra with generators z_{ij}, $1 \leq i \leq 2$, $1 \leq j \leq n$, and relations:

$${}^tzE_qz = F \quad \text{and} \quad zF^{-1}{}^tz = E_q^{-1}.$$

Let us write these relations explicitly:

$$z_{1i}z_{2j} - q^{-1}z_{2i}z_{1j} = \alpha_{ij}, \quad 1 \leq i, j \leq n \ ;$$

$$\sum_{i,j=1}^{n} \beta_{ij}z_{1i}z_{1j} = 0 = \sum_{i,j=1}^{n} \beta_{ij}z_{2i}z_{2j} \ ;$$

$$\sum_{i,j=1}^{n} \beta_{ij} z_{1i} z_{2j} = -q \quad ; \quad \sum_{i,j=1}^{n} \beta_{ij} z_{2i} z_{1j} = 1.$$

Multiplying the first relation by β_{ij}, summing over i and j, using the third relation and the identity $\mathrm{tr}(F^t F^{-1}) = -q - q^{-1}$, we see that the last relation is a consequence of the other ones.

Let us order the set $\{1, \dots, n\} \times \{1, \dots, n\}$ lexicographically. Take (u, v) the maximal element such that $\beta_{uv} \neq 0$. Since the matrix (β_{ij}) is invertible, we have $u = n$, then since $\beta_{nn} = 0$, we have $v < n$. We see now that $\mathcal{B}(E_q, F)$ is the universal algebra with generators z_{1i}, z_{2i}, $1 \leq i \leq n$, satisfying the relations:

$$\begin{cases} z_{2i} z_{1j} = q(z_{1i} z_{2j} - \alpha_{ij}) \ , \quad 1 \leq i, j \leq n \quad ; \\ z_{1n} z_{1v} = \beta_{nv}^{-1}(-\sum_{(i,j)<(n,v)} \beta_{ij} z_{1i} z_{1j}) \quad ; \\ z_{1n} z_{2v} = \beta_{nv}^{-1}(-q - \sum_{(i,j)<(n,v)} \beta_{ij} z_{1i} z_{2j}) \quad ; \\ z_{2n} z_{2v} = \beta_{nv}^{-1}(-\sum_{(i,j)<(n,v)} \beta_{ij} z_{2i} z_{2j}). \end{cases}$$

We have now a nice presentation to use the diamond lemma [3]. We freely use the techniques and definitions involved in the diamond lemma, and in particular the simplified exposition in the book [10] (although there are a few misprints there). We endow the set $\{z_{ij},\ 1 \leq i \leq 2,\ 1 \leq j \leq n\}$, with the order induced by the lexicographic order on the set $\{1, 2\} \times \{1, \dots, n\}$, then we order the set of monomials according to their length, and finally the monomials of the same length are ordered lexicographically. It is clear that the presentation above is compatible with this order. It is also clear that there are no inclusion ambiguities. There are exactly the following overlap ambiguities:

$$(z_{2i} z_{1n}, z_{1n} z_{1v}) \ , \quad 1 \leq i \leq n \quad ; \quad (z_{2i} z_{1n}, z_{1n} z_{2v}) \ , \quad 1 \leq i \leq n \quad ;$$

$$(z_{1n} z_{2v}, z_{2v} z_{1j}) \ , \quad 1 \leq j \leq n \quad ; \quad (z_{2n} z_{2v}, z_{2v} z_{1j}) \ , \quad 1 \leq j \leq n.$$

Note that that if we had $v = n$ ($\beta_{nn} \neq 0$), there would be more ambiguities. We must check now that these ambiguities are resolvable. Let us first note some preliminary identities ($i, j \in \{1, \dots, n\}$), which hold in the free algebra $k\{z_{ij}\}_{1 \leq i,j \leq n}$:

(1) $$\sum_{(k,l)<(n,v)} \alpha_{ik} \beta_{kl} z_{1l} = \sum_{k,l} \alpha_{ik} \beta_{kl} z_{1l} - \alpha_{in} \beta_{nv} z_{1v} = z_{1i} - \alpha_{in} \beta_{nv} z_{1v}.$$

Similarly, we have:

(2) $$\sum_{(k,l)<(n,v)} \alpha_{ik} \beta_{kl} z_{2l} = z_{2i} - \alpha_{in} \beta_{nv} z_{2v} \ ;$$

(3) $$\sum_{(k,l)<(n,v)} \alpha_{lj} \beta_{kl} z_{1k} = z_{1j} - \alpha_{vj} \beta_{nv} z_{1n} \ ;$$

$$(4) \qquad \sum_{(k,l)<(n,v)} \alpha_{lj}\beta_{kl}z_{2k} = z_{2j} - \alpha_{vj}\beta_{nv}z_{2n} \; ;$$

Let us check now that the overlap ambiguities of the first family are resolvable. As usual the symbol "$\to$" means that we perform a reduction. Let $i \in \{1, \dots, n\}$. We have:

$$\begin{aligned}
&q(z_{1i}z_{2n} - \alpha_{in})z_{1v} = q(z_{1i}z_{2n}z_{1v} - \alpha_{in}z_{1v}) \to q\left(z_{1i}(q(z_{1n}z_{2v} - \alpha_{nv})) - \alpha_{in}z_{1v}\right) = \\
&= q(qz_{1i}z_{1n}z_{2v} - q\alpha_{nv}z_{1i} - \alpha_{in}z_{1v}) \\
&\to q\left(qz_{1i}\left(\beta_{nv}^{-1}(-q - \sum_{(k,l)<(n,v)} \beta_{kl}z_{1k}z_{2l})\right) - q\alpha_{nv}z_{1i} - \alpha_{in}z_{1v}\right) \\
&= q\left(-q^2\beta_{nv}^{-1}z_{1i} - q\beta_{nv}^{-1}(\sum_{(k,l)<(n,v)} \beta_{kl}z_{1i}z_{1k}z_{2l}) - q\alpha_{nv}z_{1i} - \alpha_{in}z_{1v}\right).
\end{aligned}$$

On the other hand we have:

$$\begin{aligned}
&z_{2i}\left(\beta_{nv}^{-1}(-\sum_{(k,l)<(n,v)} \beta_{kl}z_{1k}z_{1l})\right) = -\beta_{nv}^{-1}\left(\sum_{(k,l)<(n,v)} \beta_{kl}z_{2i}z_{1k}z_{1l}\right) \\
&\to \; -\beta_{nv}^{-1}\left(\sum_{(k,l)<(n,v)} \beta_{kl}(q(z_{1i}z_{2k} - \alpha_{ik}))z_{1l}\right) \\
&= -\beta_{nv}^{-1}\left(\sum_{(k,l)<(n,v)} q\beta_{kl}z_{1i}z_{2k}z_{1l} - q\sum_{(k,l)<(n,v)} \alpha_{ik}\beta_{kl}z_{1l}\right) \\
&= -\beta_{nv}^{-1}\left(\sum_{(k,l)<(n,v)} q\beta_{kl}z_{1i}z_{2k}z_{1l} - q(z_{1i} - \alpha_{in}\beta_{nv}z_{1v})\right) \text{ by (1)} \\
&\to -\beta_{nv}^{-1}\left(\sum_{(k,l)<(n,v)} q\beta_{kl}z_{1i}(q(z_{1k}z_{2l} - \alpha_{kl})) - q(z_{1i} - \alpha_{in}\beta_{nv}z_{1v})\right) \\
&= -\beta_{nv}^{-1}\left(\sum_{(k,l)<(n,v)} q^2\beta_{kl}z_{1i}z_{1k}z_{2l} - q^2(\sum_{(k,l)<(n,v)} \alpha_{kl}\beta_{kl}z_{1i}) - q(z_{1i} - \alpha_{in}\beta_{nv}z_{1v})\right) \\
&= -\beta_{nv}^{-1}q\left(\sum_{(k,l)<(n,v)} q\beta_{kl}z_{1i}z_{1k}z_{2l} + q\alpha_{nv}\beta_{nv}z_{1i} + q^2z_{1i} + z_{1i} - (z_{1i} - \alpha_{in}\beta_{nv}z_{1v})\right) \\
&= q\left(-q^2\beta_{nv}^{-1}z_{1i} - q\beta_{nv}^{-1}(\sum_{(k,l)<(n,v)} \beta_{kl}z_{1i}z_{1k}z_{2l}) - q\alpha_{nv}z_{1i} - \alpha_{in}z_{1v}\right).
\end{aligned}$$

Hence the overlap ambiguities of the first family are resolvable. Let us now study the second family of ambiguities. We have:

$$q(z_{1i}z_{2n}-\alpha_{in})z_{2v} = q(z_{1i}z_{2n}z_{2v}-\alpha_{in}z_{2v}) \to q\left(-z_{1i}\beta_{nv}^{-1}(\sum_{(k,l)<(n,v)}\beta_{kl}z_{2k}z_{2l})-\alpha_{in}z_{2v}\right)$$

$$= q\left(-\beta_{nv}^{-1}(\sum_{(k,l)<(n,v)}\beta_{kl}z_{1i}z_{2k}z_{2l})-\alpha_{in}z_{2v}\right).$$

On the other hand we have:

$$z_{2i}\left(\beta_{nv}^{-1}(-q-\sum_{(k,l)<(n,v)}\beta_{kl}z_{1k}z_{2l})\right) = \beta_{nv}^{-1}\left(-\sum_{(k,l)<(n,v)}\beta_{kl}z_{2i}z_{1k}z_{2l}-qz_{2i}\right)$$

$$\to \beta_{nv}^{-1}\left(-\sum_{(k,l)<(n,v)}\beta_{kl}q(z_{1i}z_{2k}-\alpha_{ik})z_{2l}-qz_{2i}\right)$$

$$= \beta_{nv}^{-1}q\left(-\sum_{(k,l)<(n,v)}\beta_{kl}z_{1i}z_{2k}z_{2l}+\sum_{(k,l)<(n,v)}\alpha_{ik}\beta_{kl}z_{2l}-z_{2i}\right)$$

$$= \beta_{nv}^{-1}q\left(-\sum_{(k,l)<(n,v)}\beta_{kl}z_{1i}z_{2k}z_{2l}+z_{2i}-\alpha_{in}\beta_{nv}z_{2v}-z_{2i}\right) \quad \text{by (2)}$$

$$= q\left(-\beta_{nv}^{-1}(\sum_{(k,l)<(n,v)}\beta_{kl}z_{1i}z_{2k}z_{2l})-\alpha_{in}z_{2v}\right).$$

Hence the ambiguities in the second family are resolvable. The resolvability of the ambiguities of the third and fourth families is shown is the same way, using the identities (3) and (4) respectively. This is left to the reader. Since all ambiguities are resolvable and our order is compatible with the presentation, we can use the diamond lemma [3]: the set of reduced monomials (i.e. those invariant under all reductions) is a basis of $\mathcal{B}(E_q,F)$. In particular the reduced monomials z_{1i}, $1 \leq i \leq n$, are linearly independent. This shows that $\mathcal{B}(E_q,F)$ is a non-zero vector space, and concludes the proof of Proposition 3.4. □

Let $E \in GL(m)$ and $F \in GL(n)$. Let us prove now that $\mathcal{B}(E,F)$ is non-zero when $\mathrm{tr}(E^tE^{-1}) = \mathrm{tr}(F^tF^{-1})$.

Let $q \in k^*$ be such that $q^2+\mathrm{tr}(E^tE^{-1})q+1=0$. Similarly as in Section 3, we have an algebra morphism $\delta : \mathcal{B}(E,F) \longrightarrow \mathcal{B}(E,E_q)\otimes\mathcal{B}(E_q,F)$ defined by

$$\delta(z_{ij}) = \sum_{k=1}^{2} v_{ik}\otimes w_{kj},\ 1\leq i\leq m, 1\leq j\leq n,$$

where (v_{ik}) and (w_{kj}) denote the generators of $\mathcal{B}(E,E_q)$ and $\mathcal{B}(E_q,F)$ respectively. Then by the proof of Proposition 3.4, the elements w_{kj} are linearly independent, and since the algebra morphism $\phi : \mathcal{B}(E, E_q) \longrightarrow \mathcal{B}(E_q, E)^{\rm op}$ of Section 3 is an isomorphism, the elements v_{ik} are also linearly independent. Hence $\delta(z_{ij}) \neq 0$, and it follows that $\mathcal{B}(E, F)$ is a non-zero algebra.

5 $SL(2)$-deformations

In this section k will be an algebraically closed field of characteristic zero. This section is essentially devoted to the proof of Theorem 1.2. We also determine the isomorphic classification of the Hopf algebras $\mathcal{B}(E)$.

Let us first recall the concept of representation semi-ring of a Hopf algebra. Let A be a cosemisimple Hopf algebra. The representation ring of A is defined to be the Grothendieck group of the category $\mathrm{Comod}_f(A)$: $R(A) = K_0(\mathrm{Comod}_f(A))$. It is a free abelian group with a basis formed by the isomorphism classes of simple (irreducible) comodules. The monoidal structure of $\mathrm{Comod}_f(A)$ induces a ring structure on $R(A)$. The isomorphism class of a finite-dimensional A-comodule V is denoted by $[V]$. The representation semi-ring of A is now defined to be

$$R^+(A) = \{\sum_i a_i[V_i] \in R(A),\ a_i \geq 0, V_i \in \mathrm{Comod}_f(A)\}.$$

Let B be another cosemisimple Hopf algebra and let $f : A \longrightarrow B$ be a Hopf algebra morphism. Then f induces a monoidal functor $f_* : \mathrm{Comod}_f(A) \to \mathrm{Comod}_f(B)$, and hence a semi-ring morphism $f_* : R^+(A) \longrightarrow R^+(B)$. It is not difficult to see that a semi-ring isomorphism $R^+(A) \cong R^+(B)$ induces a bijective correspondence (that preserves tensor products) between the isomorphism classes of simple comodules of A and B.

Let G be a reductive algebraic group. It is classical to say that a cosemisimple Hopf algebra A is a G-deformation if one has a semi-ring isomorphism $R^+(A) \cong R^+(\mathcal{O}(G))$. Hence Theorem 1.2 classifies $SL(2)$-deformations.

Let us state a useful folk-known result. We include a sketch of proof for the sake of completeness.

Lemma 5.1 *Let A and B be cosemisimple Hopf algebras and let $f : A \longrightarrow B$ be a Hopf algebra morphism inducing a semi-ring isomorphism $R^+(A) \cong R^+(B)$. Then f is a Hopf algebra isomorphism.*

Proof. let $f_* : \mathrm{Comod}_f(A) \to \mathrm{Comod}_f(B)$ be the induced functor. Since f induces an isomorphism $R^+(A) \cong R^+(B)$, the functor f_* transforms simple objects of $\mathrm{Comod}_f(A)$ into simple objects of $\mathrm{Comod}_f(B)$ and hence is an equivalence of categories (the categories $\mathrm{Comod}_f(A)$ and $\mathrm{Comod}_f(B)$ are semisimple). Then f is an isomorphism by Tannaka-Krein type reconstruction theorems (see e.g. [8]). $\square$.

We now recall the representation theory of $\mathcal{O}(SL_q(2))$. Our reference, especially for the root of unity case, will be the paper [11] of P. Kondratowicz and P. Podleś.

Let $q \in k^*$. We say that q is generic if q is not a root of unity or if $q \in \{\pm 1\}$.

• Let us first assume that q is generic. Then $\mathcal{O}(SL_q(2))$ is cosemisimple and has a family of non-isomorphic simple comodules $(U_n)_{n\in\mathbb{N}}$ such that

$$U_0 = k, \quad U_n \otimes U_1 \cong U_1 \otimes U_n \cong U_{n-1} \oplus U_{n+1}, \quad \dim(U_n) = n+1, \ \forall n \in \mathbb{N}^*.$$

Furthermore, any simple $\mathcal{O}(SL_q(2))$-comodule is isomorphic to one of the comodules U_n.

• Now assume that q is not generic. Let $N \geq 3$ be the order of q. Then we let

$$N_0 = \begin{cases} N & \text{if } N \text{ is odd,} \\ N/2 & \text{if } N \text{ is even.} \end{cases}$$

Then $\mathcal{O}(SL_q(2))$ is not cosemisimple. There exists families $\{V_n, \ n \in \mathbb{N}\}$, $\{U_n, \ n = 0, \dots, N_0 - 1\}$ of non-isomorphic simple comodules (except for $n = 0$ where $U_0 \cong k \cong V_0$), such that

$$V_n \otimes V_1 \cong V_1 \otimes V_n \cong V_{n-1} \oplus V_{n+1}, \quad \dim(V_n) = n+1, \ \forall n \in \mathbb{N}^*.$$

$$U_n \otimes U_1 \cong U_1 \otimes U_n \cong U_{n-1} \oplus U_{n+1}, \quad \dim(U_n) = n+1, \ n = 1, \dots, N_0 - 2.$$

The comodule $U_{N_0-1} \otimes U_1$ is not semisimple. It has a simple filtration:

$$(0) \subset U_{N_0-2} \subset Y \subset U_{N_0-1} \otimes U_1,$$

with $(U_{N_0-1} \otimes U_1)/Y \cong U_{N_0-2}$ and $Y/U_{N_0-2} \cong V_1$.

The comodules $V_n \otimes U_m \cong U_m \otimes V_n$, $n \in \mathbb{N}$, $m = 0, \dots, N_0 - 1$, are simple, and any simple $\mathcal{O}(SL_q(2))$-comodule is isomorphic with one of these comodules.

Finally there is another useful fact: the Hopf subalgebra of $\mathcal{O}(SL_q(2))$ generated by the matrix coefficients of the comodule V_1 is cosemisimple and is isomorphic with $\mathcal{O}_t(SL(2))$, for $t = q^{N_0^2} = \pm 1$.

Let $E \in GL(m)$, $m \geq 2$. Let $n \in \mathbb{N}$. We denote by U_n^E and V_n^E the simple $\mathcal{B}(E)$-comodules corresponding to the simple $\mathcal{O}(SL_q(2))$-comodules U_n and V_n (for q as in Theorem 1.1).

Here is a useful lemma:

Lemma 5.2 *Let $E \in GL(m)$ and $F \in GL(n)$. Let $H : \mathrm{Comod}_f(\mathcal{B}(E)) \to \mathrm{Comod}_f(\mathcal{B}(F))$ be an equivalence of monoidal categories. Then $H(U_1^E) \cong U_1^F$. If $\mathcal{B}(E)$ is cosemisimple, then $H(U_n^E) \cong U_n^F$, $\forall n \in \mathbb{N}$. If $\mathcal{B}(E)$ is not cosemisimple, then $H(U_n^E) \cong U_n^F$, $\forall n < N_0 - 1$, and $H(V_n^E) \cong V_n^F$, $\forall n \in \mathbb{N}$.*

Proof. Let us first assume that $\mathcal{B}(E)$ is cosemisimple. One can show by induction that

$$U_k^E \otimes U_l^E \cong U_{|k-l|}^E \oplus U_{|k-l|+2}^E \oplus \dots \oplus U_{k+l}^E \text{ for } k, l \in \mathbb{N}.$$

Hence U_1^E is the only simple $\mathcal{B}(E)$-comodule W such that $W \otimes W$ is the direct sum of two simple comodules. It follows that $H(U_1^E) \cong U_1^F$, and then an easy induction, using the fusion rule $U_1^E \otimes U_n^E \cong U_{n-1}^E \oplus U_{n+1}^E$, shows that $H(U_n^E) \cong U_n^F$, $\forall n \in \mathbb{N}$.

Now assume that $\mathcal{B}(E)$ is not cosemisimple. The $\mathcal{B}(F)$-comodule $H(V_1^E)^{\otimes k}$ must be semisimple for all $k \in \mathbb{N}$, and hence we have $H(V_1^E) \cong V_i^F$ for some i. By the cosemisimple case we have $H(V_1^E) \cong V_1^F$ and $H(V_n^E) \cong V_n^F$, $\forall n \in \mathbb{N}$. Now pick Z a simple $\mathcal{B}(E)$-comodule such that $H(Z) \cong U_1^F$. We have $Z \cong U_i^E \otimes V_j^E$ for some positive integers $i \leq N_0 - 1$ and j. Hence $H(U_i^E) \otimes V_j^F \cong U_1^F$. It is then clear that $j = 0$ and $H(U_i^E) \cong U_1^F$. Now note that since $H(U_k^E) \otimes H(U_1^E) \cong H(U_{k-1}^E) \oplus H(U_{k+1}^E)$, we have

$$\dim(H(U_1^E)) < \dim(H(U_2^E)) < \ldots < \dim(H(U_{N_0-1}^E)).$$

Then if $i > 1$, we have $\dim(H(U_1^E)) < \dim(U_1^F)$. But another glance at the fusion rules shows that U_1^F is the simple $\mathcal{B}(F)$-comodule of the smallest dimension. Hence $i = 1$ and $H(U_1^E) \cong U_1^F$. Another easy induction now shows that $H(U_n^E) \cong U_n^F$, for $n \in \{0, \ldots, N_0 - 1\}$. $\square$

We now present the isomorphic classification of the Hopf algebras $\mathcal{B}(E)$.

Theorem 5.3 *Let $E \in GL(m)$ and $F \in GL(n)$. The Hopf algebras $\mathcal{B}(E)$ and $\mathcal{B}(F)$ are isomorphic if and only if $m = n$ and there exists $M \in GL(m)$ such that $F = {}^tMEM$.*

Proof. We denote by (a_{ij}) and (b_{ij}) the respective generators of $\mathcal{B}(E)$ and $\mathcal{B}(F)$, and by a and b the corresponding matrices. By the construction of the categorical equivalence of Theorem 1.1 (see [16] and [15]), the elements a_{ij} and b_{ij} are the matrix coefficients of the comodules U_1^E and U_1^F respectively.

Let $f : \mathcal{B}(E) \longrightarrow \mathcal{B}(F)$ be a Hopf algebra isomorphism and let $f_* : \mathrm{Comod}_f(\mathcal{B}(E)) \longrightarrow \mathrm{Comod}_f(\mathcal{B}(F))$ be the induced equivalence of monoidal categories. By Lemma 5.2, we have $f_*(U_1^E) \cong U_1^F$. Hence $m = n$ and there exists $P \in GL(m)$ such that $f(a) = PbP^{-1}$. But we must have $f(E^{-1}{}^taEa) = I$, and hence $b^{-1} = ({}^tPEP)^{-1}{}^tb({}^tPEP)$. On the other hand $b^{-1} = S(b) = F^{-1}{}^tbF$. Since the elements b_{ij} are linearly independent (the comodule U_1^F is simple), it follows that there exists $\lambda \in k^*$ such that $F = \lambda{}^tPEP$, and we can take $M = \sqrt{\lambda}P$.

The converse assertion is Proposition 2.3. $\square$

Remark 5.4 *The proof of Theorem 5.3 also shows that the automorphism group of the Hopf algebra $\mathcal{B}(E)$ is isomorphic with the group $G_E = \{P \in GL(m) \mid {}^tPEP = E\}/\{\pm I\}$.* $\square$

Proof of Theorem 1.2.

The proof follows closely some part of the proof of theorem 3.2 in [14]. Let A be a cosemisimple Hopf algebra with $R^+(A) \cong R^+(\mathcal{O}(SL(2))$. Let us denote by U_n^A, $n \in \mathbb{N}$, the simple A-comodules (with the same conventions as before). We have $U_1^A \otimes U_1^A \cong k \oplus U_2^A$. Hence the A-comodule U_1^A is self-dual : there exists A-comodule morphisms $e : U_1^A \otimes U_1^A \longrightarrow k$ and $\delta : k \longrightarrow U_1^A \otimes U_1^A$ such that $(e \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) = \mathrm{id}$ and $(\mathrm{id} \otimes e) \circ (\delta \otimes \mathrm{id}) = \mathrm{id}$. These equations show that the bilinear form induced by e is non-degenerate. Thus by Proposition 2.2 there exists $E \in GL(m)$ (with $m = \dim(U_1^A)$) and a Hopf algebra morphism $f : \mathcal{B}(E) \longrightarrow A$ such that $f_*(U_1^E) = U_1^A$.

First assume that $\mathcal{B}(E)$ is cosemisimple. Then using the fusion rules $U_1^E \otimes U_n^E \cong U_{n-1}^E \oplus U_{n+1}^E$, an easy induction shows that $f_*(U_n^E) \cong U_n^A$, $\forall n \in \mathbb{N}$. Hence f induces a semi-ring isomorphism $R^+(\mathcal{B}(E)) \cong R^+(A)$, and is an isomorphism by Lemma 5.1.

Now assume that $\mathcal{B}(E)$ is not cosemisimple. An induction also shows that $f_*(U_n^E) \cong U_n^A$, $\forall n \in \{0, \dots N_0 - 1\}$. Then we have

$$f_*(U_{N_0-1}^E \otimes U_1^E) \cong U_{N_0-1}^A \otimes U_1^A \cong U_{N_0-2}^A \oplus U_{N_0}^A.$$

On the other hand using the simple filtration of the $\mathcal{B}(E)$-comodule $U_{N_0-1}^E \otimes U_1^E$, we have

$$f_*(U_{N_0-1}^E \otimes U_1^E) \cong U_{N_0-2}^A \oplus f_*(V_1^E) \oplus U_{N_0-2}^A.$$

This contradicts the uniqueness of the decomposition a semisimple comodule into a direct sum of simple comodules.

Thus $\mathcal{B}(E)$ is cosemisimple, any element $q \in k^*$ such that $q^2 + \mathrm{tr}(E^tE^{-1})q + 1 = 0$ is generic, and f is an isomorphism. The last assertion in Theorem 1.2 is a direct consequence of Theorem 5.3. $\square$

6 CQG algebra structure on $\mathcal{B}(E)$

In this section $k = \mathbb{C}$. We determine the possible Hopf $*$-algebra structures and CQG algebra structures on $\mathcal{B}(E)$.

Let us recall that a Hopf $*$-algebra is a Hopf algebra A, which is also a $*$-algebra and such that the comultiplication is a $*$-homomorphism. If $a = (a_{ij}) \in M_n(A)$ is a matrix with coefficients in A, the matrix (a_{ij}^*) is denoted by $\overline{a}$, while ${}^t\overline{a}$, the transpose matrix of $\overline{a}$, is denoted by a^*. The matrix a is said to be unitary if $a^*a = I = aa^*$. Recall [10] that a Hopf $*$-algebra A is said to be a CQG algebra if for every finite-dimensional A-comodule with associate matrix of coefficients $a \in M_n(A)$, there exist $K \in GL(n)$ such that the matrix KaK^{-1} is unitary. A CQG algebra may be seen as the algebra of representative functions on a compact quantum group.

Proposition 6.1 *Let $E \in GL(m)$, and denote by a_{ij}, $1 \le i,j \le m$, the generators of $\mathcal{B}(E)$.*
1) The Hopf algebra $\mathcal{B}(E)$ admits a Hopf $*$-algebra structure if and only if there exists $M \in GL(m)$ such that

$$(\star) \quad {}^tME^*M = E, \quad \overline{M}M = \lambda I, \quad \text{for some } \lambda \in \mathbb{R}^*.$$

The $*$-structure of $\mathcal{B}(E)$ is then defined by $\overline{a} = MaM^{-1}$. The corresponding Hopf $*$-algebra is denoted by $\mathcal{B}(E)_M$. If N is another matrix satisfying the conditions $(\star)$, the Hopf $*$-algebras $\mathcal{B}(E)_M$ and $\mathcal{B}(E)_N$ are isomorphic if and only if there exists $P \in GL(m)$ such that

$$E = {}^tPEP, \quad MP = \gamma\overline{P}N \quad \text{for some } \gamma \in \mathbb{C}^*.$$

2) Let $M \in GL(m)$ satisfying the conditions $(\star)$. Then the Hopf $*$-algebra $\mathcal{B}(E)_M$ is a CQG algebra if and only if there exist $\mu \in \mathbb{C}^*$ such that the matrix $\mu {}^tM^{-1}E$ is positive.

Proof. We use the notations of Section 5: let U_1^E be the fundamental comodule of $\mathcal{B}(E)$, with $(a_{ij})_{1\leq i,j\leq m}$ as matrix coefficients.
1) Let us first assume that $\mathcal{B}(E)$ admits a Hopf $*$-algebra structure. Then by the arguments of Lemma 5.2, we have $\overline{U_1^E} \cong U_1^E$, where $\overline{U_1^E}$ denotes the conjugate comodule of $\mathcal{B}(E)$. Hence there exists $M \in GL(m)$ such that $\overline{a} = MaM^{-1}$. Now we have $\overline{E^{-1}{}^t a E a} = \overline{I} = I$, but

$$\overline{E^{-1}{}^t a E a} = {}^t({}^t(\overline{Ea}){}^t(\overline{E^{-1}{}^t a})) = {}^t({}^t(\overline{E}MaM^{-1}){}^t(\overline{E}^{-1}{}^t M^{-1}{}^t a {}^t M)) = {}^t({}^t M^{-1}{}^t a {}^t M E^* MaM^{-1}{E^*}^{-1}),$$

and hence $(M^{-1}{E^*}^{-1}{}^t M^{-1}){}^t a({}^t ME^*M)a = I$. By the linear independence of the elements a_{ij}, we have $E = \alpha {}^t ME^*M$ for some $\alpha \in \mathbb{C}^*$, and up to a normalization by $\sqrt{\alpha}$, we can assume that $E = {}^t ME^*M$. Now we have $a = \overline{\overline{a}} = \overline{MaM^{-1}} = \overline{M}MaM^{-1}\overline{M}^{-1}$. Hence by the linear independence of the elements a_{ij}, we have $\overline{M}M = \lambda I$ for some $\lambda \in \mathbb{R}^*$.

Conversely, if $M \in GL(m)$ satisfies the conditions $(\star)$, it is easy to check, using the computations already done, that one defines a Hopf $*$-algebra structure on $\mathcal{B}(E)$ by letting $\overline{a} = MaM^{-1}$.

Let $M, N \in GL(m)$ satisfying the conditions $(\star)$, and let $\phi : \mathcal{B}(E)_M \longrightarrow \mathcal{B}(E)_N$ be a Hopf $*$-algebra isomorphism. By Theorem 5.3 and its proof there exists P in $GL(m)$ such that $E = {}^t PEP$ and $\phi(a) = PaP^{-1}$. But we have

$$\phi(\overline{a}) = \phi(MaM^{-1}) = MPaP^{-1}M^{-1} \quad \text{and} \quad \overline{\phi(a)} = \overline{P}NaN^{-1}\overline{P}^{-1},$$

which implies that $MP = \gamma\overline{P}N$, for some $\gamma \in \mathbb{C}^*$. It is not difficult to prove the converse assertion using the above considerations.
2) Assume that $\mathcal{B}(E)_M$ is a CQG algebra. Then there exists $K \in GL(m)$ such that the matrix KaK^{-1} is unitary, i.e.

$$(KaK^{-1})^*(KaK^{-1}) = I = (KaK^{-1})(KaK^{-1})^*,$$

and some easy computations show that there exists $\mu \in \mathbb{C}^*$ such that ${}^t MK^*K = \mu E$, which means that $\mu {}^t M^{-1} E$ is a positive matrix.

Conversely, if there exists $\mu \in \mathbb{C}^*$ such that $\mu {}^t M^{-1} E$ is a positive matrix, then there exists $K \in GL(m)$ such that $\mu {}^t M^{-1} E = K^*K$. It is a direct computation to show that the matrix KaK^{-1} is unitary. The algebra $\mathcal{B}(E)$ is generated by the elements a_{ij}, and hence it follows from [10], Proposition 28, p. 417, that $\mathcal{B}(E)$ is a CQG algebra. $\square$

When $\overline{E}E = rI$, with $r \in \mathbb{R}$, we can take $M = r^{-\frac{1}{2}}{}^t E$, and we get the universal CQG algebras constructed by A. Van Daele and S. Wang ([17], Theorem 1.4), denoted $\mathcal{B}_u(E)$ in [19]. The isomorphic classification of these CQG algebras was presented in [19].

If there does not exist any matrix M satisfying the conditions of Proposition 6.1, it is still possible to define a CQG algebra $\mathcal{B}_u(E)$ [17]. However the fundamental comodule is no longer irreducible. It is shown in [19] that $\mathcal{B}_u(E)$ is isomorphic to a free product of universal CQG algebras ([18]).

References

[1] T. Banica, Théorie des représentations du groupe quantique compact libre $O(n)$, C.R. Acad. Sci. Paris 322, Série I (1996), 241-244.

[2] T. Banica, A reconstruction result for the R-matrix quantizations of $SU(N)$, preprint 1998.

[3] G.M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978), 178-218.

[4] A. Bruguières, Dualité tannakienne pour les quasi-groupoïdes quantiques, Comm. Algebra 25, No.3 (1997), 737-767.

[5] M. Dubois-Violette, G. Launer, The quantum group of a non-degenerate bilinear form, Phys. Lett. B 245, No.2 (1990), 175-177.

[6] D. Gurevich, Algebraic aspects of the quantum Yang-Baxter equation, Leningrad Math. J. 2, No.4 (1991) 801-828.

[7] Phung Ho Hai, On matrix quantum groups of type A_n, International J. Math. 11, No.9 (2000), 1115-1146.

[8] A. Joyal, R. Street, An introduction to Tannaka duality and quantum groups, Lecture Notes in Math. 1488, pp 413-492, Springer-Verlag, 1991.

[9] C. Kassel, *Quantum groups*, GTM 155, Springer, 1995.

[10] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer, 1997.

[11] P. Kondratowicz, P. Podleś, On representation theory of quantum $SL_q(2)$ at roots of unity, Banach Center Publ. 40 (1997), 223-248.

[12] S. Montgomery, *Hopf algebras and their actions on rings*, Amer. Math. Soc., Providence, 1993.

[13] C. Ohn, Quantum $SL(3,\mathbb{C})$ with classical representation theory, J. Algebra 213, No.2 (1999), 721-756.

[14] P. Podleś, E. Müller, Introduction to quantum groups, Rev. Math. Phys. 10 (1998), 511-551.

[15] P. Schauenburg, Hopf bigalois extensions, Comm. Algebra 24, No. 12 (1996), 3797–3825.

[16] K.H. Ulbrich, Fiber functors on finite dimensional comodules, Manuscripta Math. 65 (1989), 39-46.

[17] A. Van Daele, S. Wang, Universal quantum groups, Intern. J. Math. 7 (1996), 255-263.

[18] S. Wang, Free products of compact quantum groups, Comm. Math. Phys. 167 (1995), 671-692.

[19] S. Wang, Structure and isomorphic classification of compact quantum groups $A_u(Q)$ and $B_u(Q)$, J. Oper. Theory, to appear.

[20] S.L. Woronowicz, New quantum deformation of $SL(2,\mathbb{C})$. Hopf algebra level, Rep. Math. Phys. 30, No.2 (1991), 259-269.