Hopf-Galois Systems

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Abstract

We introduce the concept of Hopf-Galois system, a reformulation of the notion of Galois extension of the base field for a Hopf algebra. The main feature of our definition is a generalization of the antipode of an ordinary Hopf algebra. We present several examples which indicate that although our axiomatic is slightly more complicated than the classical one, it is also more natural and easier to handle with. The main application of Hopf-Galois systems is the construction of monoidal equivalences between comodule categories.

Keywords: Hopf-Galois extension, monoidal equivalence of comodule categories.

Introduction

We introduce the concept of Hopf-Galois system, a reformulation of the notion of Galois extension of the base field for a Hopf algebra. Our motivation for such a definition is to provide a natural way to construct monoidal equivalences between categories of comodules over Hopf algebras.

Let A and B be Hopf algebras (over a field k). Recall [16] that a non-zero algebra Z is said to be an A-B-biGalois extension if Z is an A-B-bicomodule algebra such that two linear maps $\kappa_l : Z \otimes Z \longrightarrow A \otimes Z$ and $\kappa_r : Z \otimes Z \longrightarrow Z \otimes B$ are bijective (see Section 1). A useful theorem of P. Schauenburg [16] brings interest for biGalois extensions: the comodule categories over A and B are monoidally equivalent if and only if there exists an A-B-biGalois extension.

In [3, 5] we constructed some examples of biGalois extensions. It is more or less clear in these papers that, when checking that the maps κ_l and κ_r are bijective, one uses the concept of Hopf-Galois system introduced in the present paper. In fact the construction of a Hopf-Galois system seems to be the easiest and most natural way to get a biGalois extension. Here is a related and well-known situation. Consider a bialgebra A. Then A is a Hopf algebra if only if the the map

$$\kappa_l : A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes m} A \otimes A$$

is bijective (this is well-know, e.g. to multiplier Hopf algebraists [23]). In concrete examples, it is much more desirable to require the existence of an antipode, although the axiomatic is slightly more involved. We adopt the same philosophy for Galois extensions.

A Hopf-Galois system consists of four non-zero algebras (A, B, Z, T). The algebras A and B are bialgebras, and Z is assumed to be an A-B-bicomodule algebra. There are also algebra morphisms $\gamma : A \longrightarrow Z \otimes T$ and $\delta : B \longrightarrow T \otimes Z$ with some associativity conditions. Finally there is a linear map $S : T \longrightarrow Z$ playing the role of an antipode. In fact the first axioms are closely related to the ones of a set of pre-equivalence data of M. Takeuchi (Definition 2.3 in [19]), the main new feature being the generalized antipode. See Section 1 for the details. The easiest way to understand the axioms is to see a Hopf-Galois system as the dual object of a groupoid with two objects, with some structures forgotten. In fact a Hopf-Galois system is always associated to a pair of objects in the groupoid of fibre functors over the comodules of a Hopf algebra. We show (Theorem 1.2) that if (A, B, Z, T) is a Hopf-Galois system, then Z is an A-B-biGalois extension. Conversely, starting from a Galois extension, it is possible to reconstruct a Hopf-Galois system.

The axiomatic of a Hopf-Galois system is more complicated than the one of an A-Galois extension or of an A-B-biGalois extension. But in a completely parallel way to the observation concerning bialgebras and Hopf algebras, it is also very natural and easy to handle with when dealing with concrete examples. In fact when one suspects an algebra to be an A-Galois extension, it is not difficult to guess what the whole Hopf-Galois system will be. We present several examples that, we hope, will convince the reader.

It is quite possible that the axiomatic of Hopf-Galois systems was already known to some experts. It is more or less implicit in [16], with the notation Z^{-1} for the fourth algebra. On the other hand we feel that it should be useful to have written down the axiomatic completely, especially in view of applications in representation theory.

Our work is organized as follows. In Section 1 we give the full definition of a Hopf-Galois system, and we show that such a system always gives rise to a biGalois extension. We also discuss the reconstruction of a Hopf-Galois system from a Galois extension. Since we have decided in this paper to concentrate on examples and applications rather than on theoretical aspects of Hopf-Galois systems, the other sections are devoted to examples. We study the following families of examples.

- Hopf-Galois systems associated to 2-cocycles, with emphasis on the function algebra on the symmetric group, and some examples generalizing those of [3] are presented.
- Hopf-Galois systems for the Hopf algebra of a non-degenerate bilinear form [7, 5].
- Hopf-Galois systems for universal cosovereign Hopf algebras [4]. This gives some improvements on the known results [2] concerning the corepresentation theory of these Hopf algebras.
- Hopf-Galois systems for free Hopf algebras generated by matrix coalgebras [18].

Notations and conventions. Throughout this paper k denotes a commutative field. The reader is assumed to be familiar with Hopf algebras, their modules, comodules, comodule algebras [13]. We also assume familiarity with monoidal categories, monoidal functors [10, 9]. The monoidal category of comodules (resp. finite-dimensional comodules) over a bialgebra A is denoted by $\mathrm{Comod}(A)$ (resp. $\mathrm{Comod_f}(A)$). The monoidal category of finite-dimensional vector spaces over k is denoted by $\mathrm{Vect_f}(k)$.

1 Definition and basic results

We present the formal definition of a Hopf-Galois system. We work in the monoidal category of vector spaces over k, but it is clear that our definition still makes sense in any braided monoidal category, and that Theorem 1.2 is valid in such a category. Let us first recall the language of Galois extensions for Hopf algebras (see [13] for a general perspective).

Let A be a Hopf algebra. A left A-Galois extension (of k) is a non-zero left A-comodule algebra Z such that the linear map κ_l defined by the composition

$$\kappa_l : Z \otimes Z \xrightarrow{\alpha \otimes 1_Z} A \otimes Z \otimes Z \xrightarrow{1_A \otimes m_Z} A \otimes Z$$

where α is the coaction of A and m_Z is the multiplication of Z, is bijective.

Similarly, a right A-Galois extension is a non-zero right A-comodule algebra Z such that the linear map κ_r defined by the composition

$$\kappa_r : Z \otimes Z \xrightarrow{1_Z \otimes \beta} Z \otimes Z \otimes A \xrightarrow{m_Z \otimes 1_A} Z \otimes A$$

where β is the coaction of A, is bijective.

Let A and B be Hopf algebras. An algebra Z is said to be an A-B-bigalois extension [16] if Z is both a left A-Galois extension and a right B-Galois extension, and if Z is an A-B-bicomodule.

Definition 1.1 *A Hopf-Galois system consists of four non-zero algebras (A, B, Z, T), with the following axioms.*

(HG1) *The algebras A and B are bialgebras.*

(HG2) *The algebra Z is an A-B-bicomodule algebra.*

(HG3) *There are algebra morphisms $\gamma : A \longrightarrow Z \otimes T$ and $\delta : B \longrightarrow T \otimes Z$ such that the following diagrams commute:*

$$\begin{CD} Z @>{\alpha}>> A \otimes Z \\ @V{\beta}VV @VV{\gamma \otimes 1_Z}V \\ Z \otimes B @>{1_Z \otimes \delta}>> Z \otimes T \otimes Z \end{CD} \qquad \begin{CD} A @>{\Delta_A}>> A \otimes A \\ @V{\gamma}VV @VV{1_A \otimes \gamma}V \\ Z \otimes T @>{\alpha \otimes 1_T}>> A \otimes Z \otimes T \end{CD} \qquad \begin{CD} B @>{\Delta_B}>> B \otimes B \\ @V{\delta}VV @VV{\delta \otimes 1_B}V \\ T \otimes Z @>{1_T \otimes \beta}>> T \otimes Z \otimes B \end{CD}$$

(HG4) *There is a linear map $S : T \longrightarrow Z$ such that the following diagrams commute:*

$$\begin{CD} A @>{\varepsilon_A}>> k @>{u_Z}>> Z \\ @V{\gamma}VV @. @AA{m_Z}A \\ Z \otimes T @>>{1_Z \otimes S}> @. Z \otimes Z \end{CD} \qquad \begin{CD} B @>{\varepsilon_B}>> k @>{u_Z}>> Z \\ @V{\delta}VV @. @AA{m_Z}A \\ T \otimes Z @>>{S \otimes 1_Z}> @. Z \otimes Z \end{CD}$$

When $A = B = Z = T$ and $\alpha = \beta = \gamma = \delta$, we just have the axioms of a Hopf algebra, the linear map S being the antipode. The axiom HG3 is very close to the axioms of a set of pre-equivalence data of M. Takeuchi ([19]), but we do not require a B-A-bicomodule structure on T. We have already mentioned the fact that the easiest way to understand the axioms of a Hopf-Galois system is to see it as the dual object of a groupoid with two objects, where some structures would have been forgotten. In fact, we have only included the axioms needed to prove the following result.

Theorem 1.2 *Let (A, B, Z, T) be a Hopf-Galois system. Then Z is an A-B-biGalois extension.*

Proof. Let $\eta_l : A \otimes Z \longrightarrow Z \otimes Z$ be the morphism defined by

$$\eta_l = (1_Z \otimes m_Z) \circ (1_Z \otimes S \otimes 1_Z) \circ (\gamma \otimes 1_Z).$$

We show that η_l is an inverse for κ_l. we have

$$\begin{aligned}
&\eta_l \circ \kappa_l = (1_Z \otimes m_Z) \circ (1_Z \otimes S \otimes 1_Z) \circ (\gamma \otimes 1_Z) \circ (1_A \otimes m_Z) \circ (\alpha \otimes 1_Z) = \\
&(1_Z \otimes m_Z) \circ (1_Z \otimes 1_Z \otimes m_Z) \circ (1_Z \otimes S \otimes 1_Z \otimes 1_Z) \circ (\gamma \otimes 1_Z \otimes 1_Z) \circ (\alpha \otimes 1_Z) = \\
&(1_Z \otimes m_Z) \circ (1_Z \otimes m_Z \otimes 1_Z) \circ (1_Z \otimes S \otimes 1_Z \otimes 1_Z) \circ (1_Z \otimes \delta \otimes 1_Z) \circ (\beta \otimes 1_Z) = \\
&(1_Z \otimes m_Z) \circ (1_Z \otimes (u_Z \circ \varepsilon_B) \otimes 1_Z) \circ (\beta \otimes 1_Z) = 1_{Z \otimes Z}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\kappa_l \circ \eta_l = (1_A \otimes m_Z) \circ (\alpha \otimes 1_Z) \circ (1_Z \otimes m_Z) \circ (1_Z \otimes S \otimes 1_Z) \circ (\gamma \otimes 1_Z) = \\
&(1_A \otimes m_Z) \circ (1_A \otimes 1_Z \otimes m_Z) \circ (1_A \otimes 1_Z \otimes S \otimes 1_Z) \circ (\alpha \otimes 1_T \otimes 1_Z) \circ (\gamma \otimes 1_Z) = \\
&(1_A \otimes m_Z) \circ (1_A \otimes m_Z \otimes 1_Z) \circ (1_A \otimes 1_Z \otimes S \otimes 1_Z) \circ (1_A \otimes \gamma \otimes 1_Z) \circ (\Delta_A \otimes 1_Z) = \\
&(1_A \otimes m_Z) \circ (1_A \otimes (u_Z \circ \varepsilon_A) \otimes 1_Z) \circ (\Delta_A \otimes 1_Z) = 1_{A \otimes Z}.
\end{aligned}$$

This proves that κ_l is an isomorphism. Similarly, we define a morphism $\eta_r : Z \otimes B \longrightarrow Z \otimes Z$ by

$$\eta_r = (m_Z \otimes 1_Z) \circ (1_Z \otimes S \otimes 1_Z) \circ (1_Z \otimes \delta),$$

and one shows in the same way that η_r is an inverse for κ_r. □

Combining Theorem 1.2 and a special case of a theorem of P. Schauenburg [17], we get the following result. It would be interesting to find a direct proof.

Corollary 1.3 *Let (A, B, Z, T) be a Hopf-Galois system. Then A and B are Hopf algebras.*

Another theorem of P. Schauenburg (Theorem 5.5 in [16]) ensures that if A and B are Hopf algebras such that there exists an A-B-biGalois extension, the comodule categories over A and B are monoidally equivalent. This theorem, combined with Theorem 1.2, yields the following result.

Corollary 1.4 *Let (A, B, Z, T) be a Hopf-Galois system. Then the categories* Comod(A) *and* Comod(B) *are monoidally equivalent.*

Let us now explain the reconstruction of a Hopf-Galois system from a Galois extension. We use Tannaka duality techniques, for which our references are [9] and [15]. We first consider the following general situation. Let $\mathcal{C}$ be a small category and let $F, G : \mathcal{C} \longrightarrow \mathrm{Vect_f}(k)$ be some functors. Following [9], Section 3, we associate a vector space $\mathsf{Hom}^{\vee}(F,G)$ to such a pair:

$$\mathsf{Hom}^{\vee}(F,G) = \bigoplus_{X \in \mathrm{ob}(\mathcal{C})} \mathrm{Hom}_k(G(X), F(X))/\mathcal{N},$$

where $\mathcal{N}$ is the linear subspace of $\bigoplus_{X \in \mathrm{ob}(\mathcal{C})} \mathrm{Hom}_k(G(X), F(X))$ generated by the elements $F(f) \circ u - u \circ G(f)$, with $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $u \in \mathrm{Hom}_k(G(Y), F(X))$. The class of an element $u \in \mathrm{Hom}_k(G(X), F(X))$ is denoted by $[X, u]$ in $\mathsf{Hom}^{\vee}(F,G)$.

This vector space represents the functor $\mathrm{Vect_f}(k) \longrightarrow \mathrm{Sets}$, $V \longmapsto \mathrm{Nat}(F, G \otimes V)$ (see [9]). Now let $K : \mathcal{C} \longrightarrow \mathrm{Vect_f}(k)$ be another functor. The universal property of $\mathsf{Hom}^{\vee}(F,G)$ gives a linear map

$$\delta^K_{F,G} : \mathsf{Hom}^{\vee}(F,G) \longrightarrow \mathsf{Hom}^{\vee}(K,G) \otimes \mathsf{Hom}^{\vee}(F,K),$$

coassociative in an obvious sense. The map $\delta^K_{F,G}$ may be described as follows. Let X in $\mathrm{ob}(\mathcal{C})$, let $\phi \in G(X)^*$, let $x \in F(X)$ and let $e_1, \dots, e_n$ be a basis of $K(X)$. Then

$$\delta^K_{F,G}([X, \phi \otimes x]) = \sum_{i=1}^{n} [X, \phi \otimes e_i] \otimes [X, e_i^* \otimes x].$$

As a particular case of the previous construction, $\mathsf{End}^{\vee}(F) := \mathsf{Hom}^{\vee}(F,F)$ is a coalgebra (the counit is induced by the trace, see [9], Section 4)

Assume now that $\mathcal{C}$ is a monoidal category and that F and G are monoidal functors. Then $\mathsf{Hom}^{\vee}(F,G)$ inherits an algebra structure, which may be described by the following formula:

$$[X, u].[Y, v] = [X \otimes Y, \widetilde{F}_{X,Y} \circ (u \otimes v) \circ \widetilde{G}^{-1}_{X,Y}],$$

where the isomorphisms $\widetilde{F}_{X,Y} : F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$ and $\widetilde{G}_{X,Y} : G(X) \otimes G(Y) \longrightarrow G(X \otimes Y)$ are part of the monoidal functors F and G. It is easy to see that the maps $\delta^K_{F,G}$ are algebra maps, and hence $\mathsf{End}^{\vee}(F)$ is a bialgebra.

Assume finally that $\mathcal{C}$ is a rigid monoidal category. This means that every object X has a left dual ([9, 10]), i.e. there exist a triplet (X^*, e_X, d_X) where $X^* \in \mathrm{ob}(\mathcal{C})$, while $e_X : X^* \otimes X \longrightarrow I$ (I is the monoidal unit of $\mathcal{C}$) and $d_X : I \longrightarrow X \otimes X^*$ are morphisms of $\mathcal{C}$ such that:

$$(1_X \otimes e_X) \circ (d_X \otimes 1_X) = 1_X \quad \text{and} \quad (e_X \otimes 1_{X^*}) \circ (1_{X^*} \otimes d_X) = 1_{X^*}.$$

The rigidity of $\mathcal{C}$ allows one to define a duality endofunctor of $\mathcal{C}$, which will be used in the proof of the following result.

Proposition 1.5 *Let $\mathcal{C}$ be a rigid monoidal category and let $F, G : \mathcal{C} \longrightarrow \mathrm{Vect_f}(k)$ be monoidal functors. Then $(\mathsf{End}^{\vee}(F), \mathsf{End}^{\vee}(G), \mathsf{Hom}^{\vee}(G,F), \mathsf{Hom}^{\vee}(F,G))$ is a Hopf-Galois system.*

Proof. We retain the notations of Definition 1.1. We put $\alpha := \delta^F_{G,F}$, $\beta := \delta^G_{G,F}$, $\gamma := \delta^G_{F,F}$ and $\delta := \delta^F_{G,G}$. It is clear that the axioms (HG1)-(HG3) are satisfied. Hence it remains to construct the linear map $S : \mathsf{Hom}^\vee(F,G) \longrightarrow \mathsf{Hom}^\vee(G,F)$. Let $X \in \mathrm{ob}(\mathcal{C})$. Then we have natural isomorphisms

$$\lambda^F_X : F(X)^* \longrightarrow F(X^*) \quad \text{and} \quad \lambda^G_X : G(X)^* \longrightarrow G(X^*)$$

such that the following diagrams commute:

$$\begin{array}{ccccc} F(X)^* \otimes F(X) & \xrightarrow{e_{F(X)}} & I & \xrightarrow{\widetilde{F}_0} & F(I) \\ \Big\downarrow{\scriptstyle \lambda^F_X \otimes 1_{F(X)}} & & & & \Big\uparrow{\scriptstyle F(e_X)} \\ F(X^*) \otimes F(X) & & \xrightarrow{\widetilde{F}_{X^*,X}} & & F(X^* \otimes X) \end{array} \qquad \begin{array}{ccccc} F(X) \otimes F(X)^* & \xleftarrow{d_{F(X)}} & I & \xrightarrow{\widetilde{F}_0} & F(I) \\ \Big\downarrow{\scriptstyle 1_{F(X)} \otimes \lambda^F_X} & & & & \Big\downarrow{\scriptstyle F(d_X)} \\ F(X) \otimes F(X^*) & & \xrightarrow{\widetilde{F}_{X,X^*}} & & F(X \otimes X^*) \end{array}$$

Let $u \in \mathrm{Hom}_k(G(X), F(X))$. We put

$$S([X,u]) = [X^*, \lambda^G_X \circ {}^t u \circ (\lambda^F_X)^{-1}].$$

It is easy to see that S is a well defined linear map. Now let $\phi \in F(X)^*$, let $x \in F(X)$ and let $e_1, \ldots, e_n$ be a basis of $G(X)$. Then we have

$$m \circ (1 \otimes S) \circ \gamma([X, \phi \otimes x]) = \sum_{i=1}^{n} [X, \phi \otimes e_i][X^*, \lambda^G_X \circ (e_i^* \otimes x) \circ (\lambda^F_X)^{-1}] =$$

$$\begin{aligned} &= \sum_{i=1}^{n} [X \otimes X^*, \widetilde{G}_{X,X^*} \circ (1_{F(X)} \otimes \lambda^G_X) \circ ((\phi \otimes e_i) \otimes (x \otimes e_i^*)) \circ (1_{F(X)} \otimes (\lambda^F_X)^{-1}) \circ \widetilde{F}^{-1}_{X,X^*}] \\ &= [X \otimes X^*, \widetilde{G}_{X,X^*} \circ (1_{F(X)} \otimes \lambda^G_X) \circ d_{G(X)} \circ (\phi \otimes x) \circ (1_{F(X)} \otimes (\lambda^F_X)^{-1}) \circ \widetilde{F}^{-1}_{X,X^*}] \\ &= [X \otimes X^*, G(d_X) \circ \widetilde{G}_0 \circ (\phi \otimes x) \circ (1_{F(X)} \otimes (\lambda^F_X)^{-1}) \circ \widetilde{F}^{-1}_{X,X^*}] \\ &= [I, \widetilde{G}_0 \circ (\phi \otimes x) \circ (1_{F(X)} \otimes (\lambda^F_X)^{-1}) \circ \widetilde{F}^{-1}_{X,X^*} \circ F(d_X)] \\ &= [I, \widetilde{G}_0 \circ (\phi \otimes x) \circ d_{F(X)} \circ \widetilde{F}_0^{-1}] = \phi(x)[I, \widetilde{G}_0 \circ \widetilde{F}_0^{-1}] = \varepsilon([X, \phi \otimes x])1. \end{aligned}$$

Since the elements $[X, \phi \otimes x]$ linearly span $\mathsf{End}^\vee(F)$, we have the commutativity of the first diagram of HG4. The commutativity of the second diagram is proved similarly. □

Remark 1.6 Proposition 1.5 generalizes [22], using exactly the same idea. More generally, Proposition 1.5 is still valid with weaker hypothesis on the target category (which we have assumed here to be $\mathrm{Vect}_\mathrm{f}(k)$): see [15]. Of course the proof is more difficult to write: see the proof of Theorem 2.4.2 in [15]. Our proof, using rank one operators in the case of $\mathrm{Vect}_\mathrm{f}(k)$, is not very elegant, but is quite straightforward.

Remark 1.7 It is easily seen that the map $S : \mathsf{Hom}^\vee(F,G) \longrightarrow \mathsf{Hom}^\vee(G,F)^{\mathrm{op}}$ constructed in the proof of Proposition 1.5 is an algebra morphism.

We can now recover a Hopf-Galois system starting from a Galois extension.

Corollary 1.8 *Let A be Hopf algebra and let Z be a left A-Galois extension. Then there exists a Hopf algebra B and an algebra T such that (A, B, Z, T) is a Hopf-Galois system.*

Proof. First consider the forgetful functor $\omega : \mathrm{Comod_f}(A) \longrightarrow \mathrm{Vect_f}(k)$. By tannakian reconstruction theorems [9, 15] the Hopf algebras A and $\mathsf{End}^\vee(\omega)$ are isomorphic: hence we identify these two Hopf algebras. Now consider the A-Galois extension Z. Following Ulbrich [21], we associate a fibre functor $\eta_Z : \mathrm{Comod_f}(A) \longrightarrow \mathrm{Vect_f}(k)$ to Z (η_Z is a monoidal, k-linear, exact and faithful functor). For an A-comodule V, we have $\eta_Z(V) = V \wedge Z$, where $V \wedge Z$ is the kernel of the double arrow:

$$\alpha_V \otimes 1_Z,\ 1_V \otimes \alpha_Z \ :\ V \otimes Z \rightrightarrows V \otimes A \otimes Z$$

($V \wedge Z$ is the cotensor product of [19]). We have an obvious monoidal natural transformation $\eta_Z \longrightarrow \omega \otimes Z$ and thus the universal property of $\mathsf{Hom}^\vee(\eta_Z, \omega)$ yields an A-colinear algebra morphism $\mathsf{Hom}^\vee(\eta_Z, \omega) \longrightarrow Z$. Since by Proposition 1.5 and Theorem 1.2 $\mathsf{Hom}^\vee(\eta_Z, \omega)$ is a left A-Galois extension, and since the category of A-Galois extensions is a groupoid [20], then $\mathsf{Hom}^\vee(\eta_Z, \omega) \cong Z$. Then, with the obvious identifications, $(\mathsf{End}^\vee(\omega), \mathsf{End}^\vee(\eta_Z), \mathsf{Hom}^\vee(\eta_Z, \omega), \mathsf{Hom}^\vee(\omega, \eta_Z))$ is the Hopf-Galois system we have announced. $\square$

Remark 1.9 In [16], Schauenburg constructs the Hopf algebra B (and the algebra T in Section 4) using different techniques. His techniques allow him to work with Hopf algebras over a ring (with a faithful flatness assumption). It is certainly possible to get the whole Hopf-Galois system using his techniques. On the other hand, when the base ring is a field, its seems that the tannakian methods used here are easier to use (it may be a question of personal taste).

Using Remark 1.7 and the proof of the last corollary, we easily have the following result, generalizing the classical fact that the antipode of a Hopf algebra is an algebra anti-morphism. Again it would be interesting to have a direct proof.

Corollary 1.10 *Let (A, B, Z, T) be a Hopf-Galois system. Then $S : T \longrightarrow Z^{\mathrm{op}}$ is an algebra morphism.*

Remark 1.11 We have done enough work to prove easily that if A and B are Hopf algebras such that there exists an A-B-biGalois extension, then the comodule categories over A and B are monoidally equivalent. This is the part of Schauenburg's Theorem 5.5 in [16] that was used to prove Corollary 1.4, certainly the most important result of the present paper. So we include a proof for the sake of completeness.

Let Z be an A-B-biGalois extension. The fibre functor $\eta_Z : \mathrm{Comod_f}(A) \longrightarrow \mathrm{Vect_f}(k)$ factorizes through $\mathrm{Comod}_f(B)$, and thus, by the universal property of $\mathsf{End}^\vee(\eta_Z)$, there exists a Hopf algebra morphism $\phi : \mathsf{End}^\vee(\eta_Z) \longrightarrow B$ such that$(1_Z \otimes \phi) \circ \alpha' = \beta$, where α' stands for the canonical coaction of $\mathsf{End}^\vee(\eta_Z)$ on Z (recall that $\eta_Z(A) = Z$). Since Z may be identified with $\mathsf{Hom}^\vee(\eta_Z, \omega)$ (proof of Corollary 1.8), it follows from Proposition 1.5 and Theorem 1.2 that Z is a right $\mathsf{End}^\vee(\eta_Z)$-Galois extension. Then we have $(1_Z \otimes \phi) \circ \kappa'_r = \kappa_r$, ($\kappa'_r$ stands for the Galois map of $\mathsf{End}^\vee(\eta_Z)$ relative to Z) and since Z is a right B-Galois extension, it follows that $1_Z \otimes \phi$ is bijective, and so is $\phi : \mathsf{End}^\vee(\eta_Z) \cong B$. Now since η_Z is a fibre functor, tannakian theorems [9, 15] ensure that $\mathrm{Comod_f}(A)$ and $\mathrm{Comod_f}(\mathsf{End}^\vee(\eta_Z))$ are monoidally equivalent. This concludes our proof since the category $\mathrm{Comod}(A)$ is the category of Ind-objects of $\mathrm{Comod_f}(A)$.

2 Hopf-Galois systems and 2-Cocycles

BiGalois extensions are associated to 2-cocycles in [16]. We review this construction in the framework of Hopf-Galois systems. After this, we study a concrete example for the function algebra on the symmetric group.

Let A be a Hopf algebra. We use Sweedler's notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$. Recall (see e.g. [6]) that a 2-cocycle is a convolution invertible linear map $\sigma : A \otimes A \longrightarrow k$ satisfying

$$\sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c) = \sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)})$$

and $\sigma(a,1) = \sigma(1,a) = \varepsilon(a)$, for all $a, b, c \in A$. The convolution inverse of σ, denoted $\bar{\sigma}$, satisfies

$$\bar{\sigma}(a_{(1)}b_{(1)}, c)\bar{\sigma}(a_{(2)}, b_{(2)}) = \bar{\sigma}(a, b_{(1)}c_{(1)})\bar{\sigma}(b_{(2)}, c_{(2)})$$

and $\bar{\sigma}(a,1) = \bar{\sigma}(1,a) = \varepsilon(a)$, for all $a, b, c \in A$.

Following [6] and [16], we associate various algebras to a 2-cocycle. First consider the algebra ${}_\sigma A$. As a vector space ${}_\sigma A = A$ and the product of ${}_\sigma A$ is defined to be

$$a_{\sigma}.b = \sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}, \quad a, b \in A.$$

We also have the algebra $A_{\bar{\sigma}}$. As a vector space we have $A_{\bar{\sigma}} = A$ and the product of $A_{\bar{\sigma}}$ is defined to be

$$a._{\bar{\sigma}}b = \bar{\sigma}(a_{(2)}, b_{(2)})a_{(1)}b_{(1)}, \quad a, b \in A.$$

Then $A_{\bar{\sigma}}$ is a left A-comodule algebra with coaction α defined by $\alpha = \Delta$. Finally we have the Hopf algebra ${}_\sigma A_{\bar{\sigma}}$ (denoted A^σ in [6]). As a coalgebra ${}_\sigma A_{\bar{\sigma}} = A$. The product of ${}_\sigma A_{\bar{\sigma}}$ is defined to be

$$a.b = \sigma(a_{(1)}, b_{(1)})\bar{\sigma}(a_{(3)}, b_{(3)})a_{(2)}b_{(2)}, \quad a, b \in A,$$

and we have the following formula for the antipode of ${}_\sigma A_{\bar{\sigma}}$:

$$S^\sigma(a) = \sigma(a_{(1)}, S(a_{(2)}))\bar{\sigma}(S(a_{(4)}), a_{(5)})S(a_{(3)}).$$

The algebra $A_{\bar{\sigma}}$ is a right ${}_\sigma A_{\bar{\sigma}}$-comodule algebra, with coaction defined by $\beta = \Delta$. In this way $A_{\bar{\sigma}}$ is an A-${}_\sigma A_{\bar{\sigma}}$-bicomodule algebra. We have the following result.

Proposition 2.1 *Let A be Hopf algebra and let $\sigma : A \otimes A \longrightarrow k$ be a 2-cocycle. Then $(A,\ {}_\sigma A_{\bar{\sigma}}, A_{\bar{\sigma}},\ {}_\sigma A)$ is a Hopf-Galois system.*

Proof. We put $\gamma = \delta = \Delta$. It is easy to see that the axiom HG3 is satisfied. Now define a linear map $\phi : {}_\sigma A \longrightarrow A_{\bar{\sigma}}$ by $\phi(a) = \sigma(a_{(1)}, S(a_{(2)}))S(a_{(3)})$, for $a \in A$. Then

$$m_{A_{\bar{\sigma}}} \circ (1_{A_{\bar{\sigma}}} \otimes \phi) \circ \gamma(a) = a_{(1)} \cdot_{\bar{\sigma}} \phi(a_{(2)}) = a_{(1)} \cdot_{\bar{\sigma}} \sigma(a_{(2)}, S(a_{(3)}))S(a_{(4)}) =$$
$$\sigma(a_{(3)}, S(a_{(4)}))\bar{\sigma}(a_{(2)}, S(a_{(5)}))a_{(1)}S(a_{(6)}) =$$
$$\bar{\sigma} * \sigma(a_{(2)}, S(a_{(3)}))a_{(1)}S(a_{(4)}) = a_{(1)}S(a_{(2)}) = \varepsilon(a)1.$$

We also have

$$m_{A_{\bar{\sigma}}} \circ (\phi \otimes 1_{A_{\bar{\sigma}}}) \circ \delta(a) = \phi(a_{(1)}) \cdot_{\bar{\sigma}} a_{(2)} = \sigma(a_{(1)}, S(a_{(2)}))S(a_{(3)}) \cdot_{\bar{\sigma}} a_{(4)} =$$
$$\sigma(a_{(1)}, S(a_{(2)}))\bar{\sigma}(S(a_{(3)}), a_{(6)})S(a_{(4)})a_{(5)} = \sigma(a_{(1)}, S(a_{(2)}))\bar{\sigma}(S(a_{(3)}), a_{(4)}) = \varepsilon(a)1,$$

by (a5) of Theorem 1.6 in [6]. Thus $(A, {}_\sigma A_{\bar{\sigma}}, A_{\bar{\sigma}}, {}_\sigma A)$ is a Hopf-Galois system. □

Let us now study an explicit example. In fact the cocycle will only be used when proving that a certain algebra is non-zero. In [3] we have constructed 2-cocycle deformations of the function algebra on the symmetric group. We generalize these results here.

Let us fix some notations. Until the end of the section k will be a characteristic zero field. We fix $m, n \in \mathbb{Z}^*$ with $m, n \geq 2$ and a primitive m-th root of unity ξ contained in k. We will work with the symmetric group S_{mn}. For a real number x, we put $E^+(x) = n$ where $n \in \mathbb{Z}$ is such that $x \in]n-1, n]$. For $i \in \{1, \dots, mn\}$, we put $i^* := E^+(\frac{i}{m}) \in \{1, \dots, n\}$.

We say that a matrix $\mathbf{p} = (p_{ij}) \in M_n(k)$ is an AST-matrix (after Artin-Schelter-Tate [1]) if $p_{ii} = 1$ and $p_{ij}p_{ji} = 1$ for all i and j. An AST-matrix is said to be of order m if $p_{ij}^m = 1$ for all i and j. The trivial AST-matrix (i.e. $p_{ij} = 1$ for all i and j) is denoted by $\mathbf{1}$.

Let $\mathbf{p} \in M_n(k)$ be an AST-matrix of order m. Let $i, j, k, l \in \{1, \dots, mn\}$. We put

$$R_{ij}^{lk}(\mathbf{p}) := \delta_{i^*k^*}\delta_{j^*l^*} \sum_{r,s=0}^{m-1} \xi^{r(i-k)+s(j-l)} p_{j^*i^*}^{rs}.$$

Definition 2.2 *Let $\mathbf{p}, \mathbf{q} \in M_n(k)$ be AST matrices of order m. The algebra $\mathcal{O}_{\mathbf{q},\mathbf{p}}(S_{mn})$ is defined to be the universal algebra with generators $(x_{ij})_{1 \leq i,j \leq mn}$ and satisfying the relations:*

$$x_{ij}x_{ik} = \delta_{jk}x_{ij} \quad ; \quad x_{ji}x_{ki} = \delta_{jk}x_{ji} \quad ; \quad \sum_{l=1}^{mn} x_{il} = 1 = \sum_{l=1}^{mn} x_{li} \quad , \ 1 \leq i, j, k \leq n. \qquad (1)$$

$$\sum_{k,l} R_{ij}^{lk}(\mathbf{p})x_{\alpha l}x_{\beta k} = \sum_{k,l} R_{lk}^{\alpha\beta}(\mathbf{q})x_{li}x_{kj} \ , \ 1 \leq i, j, \alpha, \beta \leq n. \qquad (2)$$

When $\mathbf{p} = \mathbf{q}$, then it is easily seen that $\mathcal{O}_{\mathbf{p}}(S_{mn}) := \mathcal{O}_{\mathbf{p},\mathbf{p}}(S_{mn})$ is a Hopf algebra, with coproduct defined by $\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$, counit defined by $\varepsilon(x_{ij}) = \delta_{ij}$ and antipode defined by $S(x_{ij}) = x_{ji}$ (note that $R_{ij}^{lk}(\mathbf{p}) = R_{kl}^{ji}(\mathbf{p})$). Note that the relations (2) are just FRT relations [14]. If $m = 2$, the present Hopf algebras coincide with the Hopf algebras $\mathcal{O}_{\mathbf{p}}(S_{2n})$ of [3]. The algebras $\mathcal{O}_{\mathbf{q},\mathbf{p}}(S_{mn})$ will be shown to part of a Hopf-Galois system. Before we need a lemma.

Lemma 2.3 *Let $\mathbf{p} \in M_n(k)$ be an AST matrix of order m. Then $\mathcal{O}_{\mathbf{p},\mathbf{1}}(S_{mn})$ is a non-zero algebra.*

Proof. We will use an appropriate 2-cocycle. For $1 \leq i \leq n$, put $t_i = (m(i-1)+1,\dots,mi) \in S_{mn}$, and let $H = \langle t_1,\dots,t_n\rangle$ ($H \cong (\mathbb{Z}/m\mathbb{Z})^n$). Following Artin-Schelter-Tate [1], we define $\sigma_{\mathbf{p}} : H \times H \longrightarrow k^*$ to be the unique bimultiplicative map such that $\sigma_{\mathbf{p}}(t_i,t_j) = p_{ij}$ for $i < j$ and $\sigma_{\mathbf{p}}(t_i,t_j) = 1$ for $i \geq j$. Now consider the surjective Hopf algebra morphism

$$\pi : \mathcal{O}(S_{mn}) \longrightarrow k[H], \quad x_{ij} \longmapsto \frac{\delta_{i^*j^*}}{m}\sum_{k=0}^{m-1} \xi^{k(j-i)} t_{i*}^k \ , \ 1 \leq i,j,k,l \leq mn,$$

where $\mathcal{O}(S_{mn}) = \mathcal{O}_{\mathbf{1}}(S_{mn})$ is the function algebra on the symmetric group. Composing now $\pi \otimes \pi$ with the unique k-linear extension of $\sigma_{\mathbf{p}}$ to $k[H] \otimes k[H]$, we get a 2-cocycle on $\mathcal{O}(S_{mn})$, still denoted $\sigma_{\mathbf{p}}$. This is the method of construction of 2-cocycles induced by abelian subgroups of Enock-Vainerman [8]. We have

$$\sigma_{\mathbf{p}}(x_{ij},x_{kl}) = \delta_{ij}\delta_{kl} \text{ if } i^* \geq k^* \text{ and } \sigma_{\mathbf{p}}(x_{ij},x_{kl}) = \frac{\delta_{i^*j^*}\delta_{k^*l^*}}{m^2}\sum_{r,s=0}^{m-1} \xi^{r(j-i)+s(l-k)} p_{i^*k^*}^{rs} \text{ if } i^* < k^*.$$

It is then a straightforward but tedious computation to check that the generators of ${}_{\sigma_{\mathbf{p}}}\mathcal{O}(S_{mn})$ satisfy the defining relations of $\mathcal{O}_{\mathbf{p},\mathbf{1}}(S_{mn})$, and thus $\mathcal{O}_{\mathbf{p},\mathbf{1}}(S_{mn})$ is a non-zero algebra. $\square$

Proposition 2.4 *Consider $\mathbf{p},\mathbf{q} \in M_n(k)$ some AST matrices of order m. Then $(\mathcal{O}_{\mathbf{q}}(S_{mn}), \mathcal{O}_{\mathbf{p}}(S_{mn}), \mathcal{O}_{\mathbf{q},\mathbf{p}}(S_{mn}), \mathcal{O}_{\mathbf{p},\mathbf{q}}(S_{mn}))$ is a Hopf-Galois system.*

Proof. Let $\mathbf{r} \in M_n(k)$ be another AST matrix. It is straightforward to check that there exist a unique algebra morphism

$$\delta_{\mathbf{q},\mathbf{p}}^{\mathbf{r}} : \mathcal{O}_{\mathbf{q},\mathbf{p}}(S_{mn}) \longrightarrow \mathcal{O}_{\mathbf{q},\mathbf{r}}(S_{mn}) \otimes \mathcal{O}_{\mathbf{r},\mathbf{p}}(S_{mn})$$

such that $\delta_{\mathbf{q},\mathbf{p}}^{\mathbf{r}}(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$. Similarly it easy to see (using $R_{ij}^{lk}(\mathbf{p}) = R_{kl}^{ji}(\mathbf{p})$) that there exist a unique algebra isomorphism

$$\phi : \mathcal{O}_{\mathbf{p},\mathbf{q}}(S_{mn}) \longrightarrow \mathcal{O}_{\mathbf{q},\mathbf{p}}(S_{mn})^{\mathrm{op}}$$

such that $\phi(x_{ij}) = x_{ji}$. Now using $\delta_{\mathbf{q},\mathbf{p}}^{\mathbf{1}}$, Lemma 2.3 and ϕ, we see that $\mathcal{O}_{\mathbf{q},\mathbf{p}}(S_{mn})$ and $\mathcal{O}_{\mathbf{p},\mathbf{q}}(S_{mn})$ are non-zero algebras. We have defined all the necessary structural morphisms, and it is immediate to check that the axioms of a Hopf-Galois system are satisfied. $\square$

Combining Proposition 2.4 and Corollary 1.4, we get the following result.

Corollary 2.5 *Let $\mathbf{p} \in M_n(k)$ be an AST matrix of order m. Then the the category of $\mathcal{O}_{\mathbf{p}}(S_{mn})$-comodules is monoidally equivalent to the representation category of the symmetric group S_{mn}.*

3 Hopf-Galois systems for Hopf algebras of bilinear forms

In [5] we constructed Hopf biGalois extensions for the universal Hopf algebras associated to non-degenerate bilinear forms. We reconsider this construction at the Hopf-Galois system level: this makes the considerations of [5] more transparent. Note that in general, the Hopf-Galois systems we have here cannot be obtained using 2-cocycles.

Let $E \in GL_m(k)$ and let $F \in GL_n(k)$. Recall [5] that the algebra $\mathcal{B}(E,F)$ is the universal algebra with generators x_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$, and satisfying the relations

$$F^{-1}{}^t x E x = I_n \ ; \ x F^{-1}{}^t x E = I_m,$$

where x is the matrix (x_{ij}) and I_m and I_n are the identity matrices of size m and n respectively. For $E = F$ we have the Hopf algebra $\mathcal{B}(E)$ of M. Dubois-Violette and G. Launer [7].

Proposition 3.1 *Let $E \in GL_m(k)$ and let $F \in GL_n(k)$ $(m, n \geq 2)$ be such that* $\mathrm{tr}(E^tE^{-1}) = \mathrm{tr}(F^tF^{-1})$. *Then $(\mathcal{B}(E), \mathcal{B}(F), \mathcal{B}(E,F), \mathcal{B}(F,E))$ is a Hopf-Galois system.*

Proof. First the end of Section 4 in [5] ensures that $\mathcal{B}(E,F)$ is a non-zero algebra. Let $G \in GL_p(k)$. It is a direct computation to check that there exists a unique algebra morphism $\delta_{E,F}^G : \mathcal{B}(E,F) \longrightarrow \mathcal{B}(E,G) \otimes \mathcal{B}(G,F)$ such that $\delta_{E,F}^G(x_{ij}) = \sum_{k=1}^p x_{ik} \otimes x_{kj}$, $1 \leq i \leq m, 1 \leq j \leq n$. Also there exists a unique algebra isomorphism $\phi : \mathcal{B}(F,E) \longrightarrow \mathcal{B}(E,F)^{\mathrm{op}}$ such that $\phi(x) = F^{-1}{}^t x E$. In this way we have all the necessary structural maps and it is immediate to check that we indeed have a Hopf-Galois system. $\square$

Using Proposition 3.1 and Corollary 1.4, we have the following result from [5]:

Corollary 3.2 *1) Let $E \in GL_m(k)$ and let $F \in GL_n(k)$ $(m, n \geq 2)$ be such that* $\mathrm{tr}(E^tE^{-1}) = \mathrm{tr}(F^tF^{-1})$. *Then the categories* $\mathrm{Comod}(\mathcal{B}(E))$ *and* $\mathrm{Comod}(\mathcal{B}(F))$ *are monoidally equivalent.*
2) Assume that k is algebraically closed. Let $E \in GL_m(k)$ $(m \geq 2)$ and let $q \in k^*$ be such that $q^2 + \mathrm{tr}(E^tE^{-1})q + 1 = 0$. Then the categories $\mathrm{Comod}(\mathcal{B}(E))$ *and* $\mathrm{Comod}(\mathcal{O}(SL_q(2)))$ *are monoidally equivalent.*

4 Hopf-Galois systems for cosovereign Hopf algebras

Recall [4] that a Hopf algebra A is said to be cosovereign if there exists a character $\Phi \in A^*$ such that $S^2 = \Phi * \mathrm{id} * \Phi^{-1}$. The universal (or free) cosovereign Hopf algebras were constructed in [4]. We describe some of the Hopf-Galois systems associated with this class of Hopf algebras. Our constructions are certainly incomplete, but they nevertheless enable us to improve on certain known results [2] on the corepresentation theory of the universal cosovereign Hopf algebras (when $k = \mathbb{C}$).

Definition 4.1 *Let $E \in GL_m(k)$ and let $F \in GL_n(k)$. The algebra $H(E,F)$ is defined to be the universal algebra with generators u_{ij}, v_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$, and satisfying the relations*

$$u^t v = I_m = vF^t uE^{-1} \quad ; \quad {}^t v u = I_n = F^t uE^{-1} v.$$

When $E = F$, we just have the universal cosovereign Hopf algebras $H(F)$ of [4]. It is known (see Proposition 3.3 in [4]) that the Hopf algebra $H(F)$ remains unchanged, up to isomorphism, if the matrix F is multiplied by a non-zero scalar or is replaced by a conjugate matrix. Similarly, we have the following result.

Proposition 4.2 *Let $\lambda \in k^*$, let $E, P \in GL_m(k)$ and let $F, Q \in GL_n(k)$. Then $H(\lambda E, \lambda F) = H(E,F)$, and we have algebra isomorphisms $H(E,F) \cong H(PEP^{-1}, QFQ^{-1})$ and $H(E,F) \cong H({}^tE^{-1}, {}^tF^{-1})$.*

Proof. The first assertion is obvious. It is easily seen that there exists a unique algebra isomorphism $f : H(E,F) \longrightarrow H(PEP^{-1}, QFQ^{-1})$ such that $f(u) = {}^tPu^tQ^{-1}$ and $f(v) = P^{-1}vQ$. Also we have an algebra isomorphism $g : H(E,F) \longrightarrow H({}^tE^{-1}, {}^tF^{-1})$ such that $g(u) = v$ and $g(v) = EuF^{-1}$. □

The Hopf algebra structure of $H(F)$ is a particular case of the following result.

Proposition 4.3 *Let $E \in GL_m(k)$ and let $F \in GL_n(k)$. Assume that $H(E,F) \neq 0$. Then $(H(E), H(F), H(E,F), H(F,E))$ is a Hopf-Galois system.*

Proof. Let $G \in GL_p(k)$. Then it is easy to check that there exists a unique algebra morphism $\delta^G_{E,F} : H(E,F) \longrightarrow H(E,G) \otimes H(G,F)$ such that $\delta^G_{E,F}(u_{ij}) = \sum_{k=1}^p u_{ik} \otimes u_{kj}$ and $\delta^G_{E,F}(v_{ij}) = \sum_{k=1}^p v_{ik} \otimes v_{kj}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Also there is a unique algebra morphism $\phi : H(F,E) \longrightarrow H(E,F)^{\mathrm{op}}$ such that $\phi(u) = {}^tv$ and $\phi(v) = F^tuE^{-1}$. Thus with the obvious structural morphisms, we have a Hopf-Galois system. □

Of course this last result is useful only when one knows that $H(E,F)$ is non-zero. In view of the results of the preceding section, it is quite natural to think that $H(E,F)$ will be a non-zero algebra when $\mathrm{tr}(E) = \mathrm{tr}(F)$ and $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$. This problem will be studied elsewhere. There is already an interesting case where we can prove that $H(E,F) \neq 0$.

Proposition 4.4 *Let $E \in GL_m(k)$ and let $F \in GL_n(k)$ be such that $\mathrm{tr}(E) = \mathrm{tr}(F)$. Assume that there exists $G \in GL_m(k)$ and $K \in GL_n(k)$ such that $E = {}^tGG^{-1}$ and $F = {}^tKK^{-1}$. Then $H(E,F)$ is a non-zero algebra.*

Proof. It is easy to check that there exists a unique algebra morphism $f : H(E,F) \longrightarrow \mathcal{B}(G,K)$ such that $f(u) = x$ and $f(v) = {}^tGx^tK^{-1}$. We have $\mathrm{tr}(E) = \mathrm{tr}({}^tGG^{-1}) = \mathrm{tr}(F) = \mathrm{tr}({}^tKK^{-1})$, so by [5], we know that $\mathcal{B}(G,K)$ is a non-zero algebra. Since f is surjective, it is clear that $H(E,F)$ is a non-zero algebra. □

Let $q \in k^*$. In the next result, we consider the matrix $F_q = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \in GL_2(k)$, and we put $H_q := H(F_q)$.

Corollary 4.5 *1) Let $E \in GL_m(k)$ and let $F \in GL_n(k)$ be such that* $\mathrm{tr}(E) = \mathrm{tr}(F)$. *Assume that there exists $G \in GL_m(k)$ and $K \in GL_n(k)$ such that $E = {}^tGG^{-1}$ and $F = {}^tKK^{-1}$. Then the categories* $\mathrm{Comod}(H(E))$ *and* $\mathrm{Comod}(H(F))$ *are monoidally equivalent.*
2) Let $F \in GL_n(k)$. Assume that k is algebraically closed and that there exists $K \in GL_n(k)$ such that $F = {}^tKK^{-1}$. Let $q \in k^*$ be such that $q^2 - \mathrm{tr}(F)q + 1 = 0$. Then the categories $\mathrm{Comod}(H(F))$ *and* $\mathrm{Comod}(H_q)$ *are monoidally equivalent.*

Proof. The first assertion follows from Propositions 4.3-4.4 and Corollary 1.4. We have $F_q = {}^tGG^{-1}$ for the matrix $G = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$, and hence the second assertion follows from the fist one. □

Until the end of the section, we assume that $k = \mathbb{C}$. Let us recall that a Hopf $*$-algebra is a Hopf algebra A, which is also a $*$-algebra and such that the comultiplication is a $*$-homomorphism. Recall [11] that a Hopf $*$-algebra A is said to be a CQG algebra if for every finite-dimensional A-comodule with associate matrix of coefficients $a \in M_n(A)$, there exists $K \in GL_n(\mathbb{C})$ such that the matrix KaK^{-1} is unitary. A CQG algebra may be seen as the algebra of representative functions on a compact quantum group.

Let $F \in GL_n(\mathbb{C})$. We have seen in [4] (Proposition 3.6) that $H(F)$ admits a CQG algebra structure if and only if F is conjugate to a relatively positive matrix (a matrix M is said to be relatively positive if there exists $\lambda \in \mathbb{C}^*$ such that λM is a positive matrix). In this case $H(F)$ is the dense Hopf $*$-algebra of one the universal compact quantum groups introduced by A. Van Daele and S. Wang [24], and the corepresentation theory has been worked out by T. Banica [2]: the irreducible comodules are labelled by the free product $\mathbb{N} * \mathbb{N}$. We can combine Banica's results [2] and Corollary 4.5 to get the cosemisimplicity of some universal cosovereign Hopf algebras which do not admit a CQG algebra structure, as well as their corepresentation theory.

Example 4.6 Let $q, \alpha \in \mathbb{C}^*$ Consider the matrix $F = \begin{pmatrix} q & \alpha \\ 0 & q^{-1} \end{pmatrix}$. Since $H(F) = H(-F)$, we can assume that $q \neq 1$ without changing the Hopf algebra $H(F)$. The matrix F is not relatively positive, but satisfies the condition of Corollary 4.5, for $K = \begin{pmatrix} \alpha\frac{q}{1-q} & 1 \\ q & 0 \end{pmatrix}$, and hence $\mathrm{Comod}(H(F)) \cong^{\otimes} \mathrm{Comod}(H_q)$. If $q \in \mathbb{R}^*$, then H_q is a CQG algebra and we can use the results of [2].

Another example is constructed as follows. Let ξ be a primitive m-th root of unity, $m \geq 5$. Consider the diagonal matrix $F = \mathrm{Diag}(\xi, 1, \xi^{-1})$. Then F is not a relatively positive matrix, but F satisfies the condition of Corollary 4.5 (easy to check) and the solutions of $q^2 - (1+\xi+\xi^{-1})q + 1 = 0$ are real numbers. Hence we have $\mathrm{Comod}(H(F)) \cong^{\otimes} \mathrm{Comod}(H_q)$, and H_q is CQG algera since q is a real number: we can use the results of [2].

5 Hopf-Galois systems for free Hopf algebras

M. Takeuchi has constructed in [18] the free Hopf algebra generated by a coalgebra. We consider here the case of a matrix coalgebra $M_m(k)^*$, and construct the corresponding

Hopf-Galois system.

Definition 5.1 *Let $m,n \in \mathbb{N}^*$. The algebra $H(m,n)$ is defined to be the universal algebra with generators $x_{ij}^{(\alpha)}$, $1 \le i \le m, 1 \le j \le n$, $\alpha \in \mathbb{N}$, and submitted to the relations:*

$$x^{(\alpha)}\,{}^t x^{(\alpha+1)} = I_m \ , \ {}^t x^{(\alpha+1)} x^{(\alpha)} = I_n \ , \ \alpha \in \mathbb{N}.$$

When $m = n$, we have the free Hopf algebra $H(m,m) = H(m) = H(M_m(k)^*)$ of [18]. This Hopf algebra is also considered in [24], under a different notation. See [18] or [24] for the structural morphisms of the Hopf algebra $H(m)$. In fact we have the following more general result.

Proposition 5.2 *Let $m,n \geq 2$. Then $(H(m), H(n), H(m,n), H(n,m))$ is a Hopf-Galois system.*

Proof. Let us first check $H(m,n)$ is a non-zero algebra. Let $E \in GL_m(k)$ and $F \in GL_n(k)$ be such that $\mathrm{tr}(E{}^tE^{-1}) = \mathrm{tr}(F{}^tF^{-1})$. It is a direct computation to check that there exists a unique algebra morphism $f : H(m,n) \longrightarrow \mathcal{B}(E,F)$ such that

$$f(x^{(2k)}) = (E^{-1}{}^tE)^k x(F^{-1}{}^tF)^k \text{ and } f(x^{(2k+1)}) = {}^tE(E^{-1}{}^tE)^k x(F^{-1}{}^tF)^{k\,t}F^{-1}, \ k \in \mathbb{N}.$$

Thus, since f is surjective and $\mathcal{B}(E,F)$ is a non-zero algebra [5], it is clear that $H(m,n)$ is a non-zero algebra. Let $p \geq 2$. There is a unique algebra morphism $\delta_{m,n}^p : H(m,n) \longrightarrow H(m,p) \otimes H(p,n)$ such that $\delta_{m,n}^p(x_{ij}^{(\alpha)}) = \sum_{k=1}^p x_{ik}^{(\alpha)} \otimes x_{kj}^{(\alpha)}$, $1 \le i \le m, 1 \le j \le n$, $\alpha \in \mathbb{N}$. Also there is a unique algebra morphism $\phi : H(n,m) \longrightarrow H(m,n)^{\mathrm{op}}$ such that $\phi(x^{(\alpha)}) = {}^t x^{(\alpha+1)}$. Thus with the obvious structural morphisms, we have a Hopf-Galois system. $\square$

Combining Proposition 5.2 and Corollary 1.4, we have:

Corollary 5.3 *Let $m \geq 2$. Then the categories* $\mathrm{Comod}(H(m))$ *and* $\mathrm{Comod}(H(2))$ *are monoidally equivalent.*

There is also a version of free Hopf algebras with a bijective antipode, considered in [12] and [24]. It is left as an exercice to the reader, using the preceding techniques, to construct the corresponding Hopf-Galois systems.

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