Cohomological dimensions of Hopf algebras

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Question 1

Let A, B be Hopf algebras such that

$$\mathrm{Comod}(A) \simeq^{\otimes} \mathrm{Comod}(B)$$

How are their Hochschild cohomologies related ? In particular do we have $\mathrm{cd}(A) = \mathrm{cd}(B)$?

$\mathrm{cd}(A)$ is the usual Hochschild cohomological dimension :

$$\mathrm{cd}(A) = \sup\{n : \ H^n(A, M) \neq 0 \text{ for some } A-\text{bimodule } M\} \in \mathbb{N} \cup \{\infty\}$$

Strategy

(1) Consider a cohomology theory that is well-behaved with respect to this situation : Gerstenhaber-Schack cohomology.
(2) Try to compare $H^*_{\mathrm{GS}}(A, -)$ and $H^*(A, -)$.

Cohomological dimension

Let A be an algebra : $\mathrm{cd}(A)$ is the usual Hochschild cohomological dimension

$$\mathrm{cd}(A) = \sup\{n : \ H^n(A, M) \neq 0 \text{ for some } A - \text{bimodule } M\} \in \mathbb{N} \cup \{\infty\}$$

• If $A = \mathcal{O}(G)$, with G a compact Lie group, then

$$\mathrm{cd}(\mathcal{O}(G)) = \dim(G)$$

• If $A = \mathbb{C}\Gamma$, with Γ a discrete group, then $\mathrm{cd}(\mathbb{C}\Gamma) = \mathrm{cd}_{\mathbb{C}}(\Gamma)$, the cohomological dimension of Γ with coefficients $\mathbb{C}$. Recall that if Γ is finitely generated, then $\mathrm{cd}(\mathbb{C}\Gamma) = 1$ if and only if Γ has a free subgroup of finite index (Dunwoody's theorem).

• If A is a finite-dimensional Hopf algebra, then either $\mathrm{cd}(A) = 0$ (A is semisimple) or $\mathrm{cd}(A) = \infty$.

Hochschild Cohomology

If A is an algebra and M is an A-bimodule, the Hochschild cohomology groups $H^*(A, M)$ are the cohomology groups of the complex

$$0 \to \mathrm{Hom}(\mathbb{C}, M) \to \mathrm{Hom}(A, M) \to \mathrm{Hom}(A^{\otimes 2}, M) \to$$
$$\cdots \to \mathrm{Hom}(A^{\otimes n}, M) \to \mathrm{Hom}(A^{\otimes n+1}, M) \to \cdots$$

where the differential $\delta : \mathrm{Hom}(A^{\otimes n}, M) \to \mathrm{Hom}(A^{\otimes n+1}, M)$ is given by

$$\begin{aligned}\delta(f)(a_1 \otimes \cdots \otimes a_{n+1}) =& a_1 \cdot f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}\end{aligned}$$

This is not very helpful for concrete computations...
Example : if $A = \mathcal{O}(G)$, with G a compact Lie group, then $H^*(\mathcal{O}(G), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \Lambda^*(\mathfrak{g})$, where $\mathfrak{g} = \mathrm{Der}_\varepsilon(\mathcal{O}(G))$.

Hochschild Cohomology

Let A be a Hopf algebra, and let M be an A-bimodule. Recall that $H^*(A, M) \simeq \mathrm{Ext}^*_A(\mathbb{C}_\varepsilon, M')$, where M' is the right A-module defined by $m \leftarrow a = S(a_{(1)}) \cdot m \cdot a_{(2)}$.
To compute the Ext-groups, one has to :
(1) find a projective resolution of the trivial object $\mathbb{C}_\varepsilon$:

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon \to 0$$

(2) Apply the functor $\mathrm{Hom}_A(-, M')$ and consider the resulting complex

$$0 \to \mathrm{Hom}_A(P_0, M') \to \mathrm{Hom}_A(P_1, M') \to \mathrm{Hom}_A(P_2, M') \to \cdots$$

The cohomology groups $H^*(A, M)$ are isomorphic to the resulting cohomology groups, and are independent of the choice of the projective resolution.
Note that $\mathrm{cd}(A)$ is the length of the shortest possible resolution of $\mathbb{C}_\varepsilon$ by projective A-modules.

Yetter-Drinfeld modules

A is a Hopf algebra.

Definition

A (right-right) Yetter-Drinfeld module over A is a right A-comodule and right A-module V satisfying the condition, $\forall v \in V$, $\forall a \in A$,

$$(v \leftarrow a)_{(0)} \otimes (v \leftarrow a)_{(1)} = v_{(0)} \leftarrow a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

$\rightsquigarrow$ category $\mathcal{YD}_A^A$, with $\mathcal{YD}_A^A \simeq^{\otimes} \mathcal{Z}(\mathrm{Comod}(A)) \simeq^{\otimes} \mathcal{Z}(\mathrm{Mod}(A))$

Examples and definition

(1) $A_{\mathrm{coad}} := A_A$ as a right A-module, and $\mathrm{ad}_r(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)}$.
(2) More generally, if $V \in \mathrm{Comod}(A) \rightsquigarrow V \boxtimes A$, with

$$(v \otimes a) \leftarrow b = v \otimes ab, \ \alpha_{V \boxtimes A}(v \otimes a) = v_{(0)} \otimes a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

In particular $\mathbb{C} \boxtimes A = A_{\mathrm{coad}}$. A Yetter-Drinfeld module of type $V \boxtimes A$ is called *free*. A *relative projective* Yetter-Drinfeld module is a direct summand of a free one. These are projective as A-modules.

Gerstenhaber-Schack cohomology

$V \in \mathcal{YD}_A^A \rightsquigarrow H^*_{\mathrm{GS}}(A, V)$, Gerstenhaber-Schack cohomology, defined by an explicit bicomplex (originally defined in terms of Hopf bimodules).
$H^*_{\mathrm{GS}}(A, \mathbb{C}) =: H^*_b(A)$ is the bialgebra cohomology of A.

This cohomology theory was introduced in view of deformation theory. It was later used by Stefan to show that the set of isomorphism classes of semisimple cosemisimple Hopf algebras of a given dimension is finite.

Theorem (Taillefer)

$$H^*_{\mathrm{GS}}(A, V) \simeq \mathrm{Ext}^*_{\mathcal{YD}_A^A}(\mathbb{C}, V)$$

We will use this description as a definition.
We thus can, provided that $\mathcal{YD}_A^A$ has enough projective objects (this holds if A is cosemisimple), compute Gerstenhaber-Schack cohomology by using resolutions by projective objects (if A is cosemisimple, the projective Yetter-Drinfeld modules are exactly the direct summands of the free ones).

Proposition

Let A be a cosemimple Hopf algebra and let $V \in \mathcal{YD}_A^A$. The Gerstenhaber-Schack cohomology $H^*_{\mathrm{GS}}(A, V)$ is the cohomology of the complex

$$0 \to \mathrm{Hom}^A(\mathbb{C}, V) \to \mathrm{Hom}^A(A^{\boxtimes 1}, V) \to \mathrm{Hom}^A(A^{\boxtimes 2}, V) \to \cdots \to \mathrm{Hom}^A(A^{\boxtimes n}, V) \to \mathrm{Hom}^A(A^{\boxtimes n+1}, V) \to \cdots$$

where the differential $\partial : \mathrm{Hom}^A(A^{\boxtimes n}, V) \to \mathrm{Hom}^A(A^{\boxtimes n+1}, V)$ is given by

$$\begin{aligned}\partial(f)(a_1 \otimes \cdots \otimes a_{n+1}) =& \varepsilon(a_1) f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}\end{aligned}$$

Here $A^{\boxtimes n} = A^{\boxtimes n-1} \boxtimes A$ and Hom^A means morphisms of comodules.

Proof

The standard resolution of $\mathbb{C}_\varepsilon$ yields in a fact resolution of $\mathbb{C}$ by free Yetter-Drinfeld modules in the category $\mathcal{YD}_A^A$

$$\cdots \longrightarrow A^{\boxtimes n+1} \longrightarrow A^{\boxtimes n} \longrightarrow \cdots \longrightarrow A^{\boxtimes 2} \longrightarrow A^{\boxtimes 1} \longrightarrow 0$$

where each differential is given by

$$\begin{aligned} A^{\boxtimes n+1} &\longrightarrow A^{\boxtimes n} \\ a_1 \otimes \cdots \otimes a_{n+1} &\longmapsto \varepsilon(a_1)a_2 \otimes \cdots \otimes a_{n+1} + \\ &\sum_{i=1}^{n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \end{aligned}$$

Since A is cosemisimple, these are projective objects in $\mathcal{YD}_A^A$. Applying the functor $\mathrm{Hom}_{\mathcal{YD}_A^A}(-, V)$, we get the announced complex, whose cohomology is $H^*_{\mathrm{GS}}(A, V)$ (by the Ext-description). □

Theorem

Let $F : \mathrm{Comod}(A) \simeq^{\otimes} \mathrm{Comod}(B)$ *be a tensor equivalence. Then* F *extends to a tensor equivalence* $\hat{F} : \mathcal{YD}_A^A \simeq^{\otimes} \mathcal{YD}_B^B$ *such that*

1. *For any* $V \in \mathcal{YD}_A^A$, $H^*_{\mathrm{GS}}(A, V) \simeq H^*_{\mathrm{GS}}(B, \hat{F}(V))$. *In particular*

$$\mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{cd}_{\mathrm{GS}}(B)$$

2. $V \in \mathcal{YD}_A^A$ *is relative projective* $\Rightarrow \hat{F}(V) \in \mathcal{YD}_B^B$ *is relative projective. In particular, if* $\mathbf{P}_. \to \mathbb{C}$ *is a resolution of* $\mathbb{C}$ *by relative projectives in* $\mathcal{YD}_A^A$, *then* $\hat{F}(\mathbf{P}.) \to \mathbb{C}$ *is a resolution of* $\mathbb{C}$ *by relative projectives in* $\mathcal{YD}_B^B$.

Here of course we put

$$\mathrm{cd}_{\mathrm{GS}}(A) = \sup\{n : \ H^n_{\mathrm{GS}}(A, V) \neq 0 \text{ for some YD module } V\} \in \mathbb{N} \cup \{\infty\}$$

This result shows that the Hochschild cohomologies of A and B are indeed related.

Theorem

Let A be a Hopf algebra. There exists a functor

$$\begin{aligned} \mathrm{Bimod}(A) &\longrightarrow \mathcal{YD}_A^A \\ M &\longmapsto M\#A \end{aligned}$$

such that

$$H^*(A, M) \simeq H^*_{\mathrm{GS}}(A, M\#A)$$

In particular we have $\mathrm{cd}(A) \leq \mathrm{cd}_{\mathrm{GS}}(A)$.

$M\#A$ is $M \otimes A$ endowed with the Yetter-Drinfeld module structure defined by $m \otimes a \mapsto m \otimes a_{(1)} \otimes a_{(2)}$, $(m \otimes a) \leftarrow b = S(b_{(2)}).m.b_{(3)} \otimes S(b_{(1)})ab_{(4)}$.

The isomorphism is obtained (in the cosemisimple case) via an isomorphism of complexes.

This shows that the Gerstenhaber-Schack cohomology of a Hopf algebra completely determines its Hochschild cohomology.

Back to Question 1, our most general answer is

Corollary

Let A and B be Hopf algebras such that $\mathrm{Comod}(A) \simeq^{\otimes} \mathrm{Comod}(B)$. Then there exists two functors

$$F_1 : \mathrm{Bimod}(A) \to \mathcal{YD}_B^B \quad \text{and} \quad F_2 : \mathrm{Bimod}(B) \to \mathcal{YD}_A^A$$

such that for any A-bimodule M and any B-bimodule N,

$$H^*(A, M) \simeq H^*_{\mathrm{GS}}(B, F_1(M)) \quad \text{and} \quad H^*(B, N) \simeq H^*_{\mathrm{GS}}(A, F_2(N))$$

In particular we have $\max(\mathrm{cd}(A), \mathrm{cd}(B)) \leq \mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{cd}_{\mathrm{GS}}(B)$.

This leads to a new question :

Question 2

Let A be a Hopf algebra. Is it true that $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$?

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Let A be a Hopf algebra. Is it true that $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$?

• Positive answer to Question 2 $\Rightarrow$ positive answer to the last part of Question 1.

• A positive answer to Question 2 would be a natural infinite-dimensional generalization of a famous result by Larson-Radford : a finite-dimensional cosemisimple Hopf algebra is semisimple.

We have the following partial answer.

Theorem

Let A be cosemisimple of Kac type. Then $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$.

Examples

Theorem

Let $A = \mathcal{O}_q(\mathrm{SL}_2)$.

1. $\mathbb{C}$ *has a length* 3 *resolution by free Yetter-Drinfeld modules.*
2. $\mathrm{cd}(A) = 3$, *and if* q *is generic, then* $3 = \mathrm{cd}_{\mathrm{GS}}(A)$.
3. *For* q *generic, we have*

$$H_b^n(A) \simeq \begin{cases} 0 & \text{if } n \neq 0, 3 \\ \mathbb{C} & \text{if } n = 0, 3 \end{cases}$$

(Of course that $\mathrm{cd}(A) = 3$ was known since a long time)

Remark : the (non)-dimension drop.
If G is a compact Lie group of dimension n ($= \mathrm{cd}(\mathcal{O}(G))$), we have

$$H^n(\mathcal{O}(G), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \mathbb{C}$$

This is known not to hold anymore for quantum groups, for example

$$H^3(\mathcal{O}_q(\mathrm{SL}_2), {}_\varepsilon\mathbb{C}_\varepsilon) = (0)$$

if $q \neq \pm 1$. This is the so-called dimension drop. Hadfield-Krähmer have noticed that this can be repaired using appropriate coefficients

$$H^3(\mathcal{O}_q(\mathrm{SL}_2), {}_\varepsilon\mathbb{C}_{\Phi * \Phi}) \simeq \mathbb{C}$$

where Φ is the modular character of $\mathcal{O}_q(\mathrm{SL}_2)$. In Gerstenhaber-Schack cohomology, we have

$$H^3_b(\mathcal{O}_q(\mathrm{SL}_2)) = H^3_{\mathrm{GS}}(\mathcal{O}_q(\mathrm{SL}_2), \mathbb{C}) \simeq \mathbb{C}$$

so we do not have to care about "twisting" the coefficients of the trivial object : no dimension drop. Can this be further generalized ?

Let $E \in \mathrm{GL}_n(\mathbb{C})$, $n \geq 2$, and consider the algebra $\mathcal{B}(E)$ presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and relations

$$E^{-1}u^tEu = I_n = uE^{-1}u^tE,$$

where u is the matrix $(u_{ij})_{1\leq i,j\leq n}$. It has a Hopf algebra structure defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},\ \varepsilon(u_{ij}) = \delta_{ij},\ S(u) = E^{-1}u^tE$$

The Hopf algebra $\mathcal{B}(E)$, introduced by Dubois-Violette and Launer, represents the quantum symmetry group of the bilinear form associated to the matrix E. For the matrix

$$E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

we have $\mathcal{B}(E_q) = \mathcal{O}_q(\mathrm{SL}_2)$.
For $q \in \mathbb{C}^*$ satisfying $\mathrm{tr}(E^{-1}E^t) = -q - q^{-1}$, the tensor categories of comodules over $\mathcal{B}(E)$ and $\mathcal{O}_q(\mathrm{SL}_2)$ are equivalent.

For $q \in \mathbb{C}^*$ satisfying $\mathrm{tr}(E^{-1}E^t) = -q - q^{-1}$, the tensor categories of comodules over $\mathcal{B}(E)$ and $\mathcal{O}_q(\mathrm{SL}_2)$ are equivalent.

Theorem

Let $A = \mathcal{B}(E)$.

1. $\mathbb{C}$ *has a length* 3 *resolution by free Yetter-Drinfeld modules.*
2. $\mathrm{cd}(A) = 3$*, and if q is generic, then* $3 = \mathrm{cd}_{\mathrm{GS}}(A)$.
3. *For q generic, we have*

$$H_b^n(A) \simeq \begin{cases} 0 & \text{if } n \neq 0, 3 \\ \mathbb{C} & \text{if } n = 0, 3 \end{cases}$$

For $E = I_n$, the resolution was found by Collins-Haertl-Thom (without the Yetter-Drinfeld structure). It enabled them to show that the L^2-Betti numbers of the compact Hopf $*$-algebra $A_o(n) = \mathcal{B}(I_n)$ all vanish (the vanishing of the first L^2-Betti number was due to Vergnioux).

Theorem

Let $A = \mathcal{O}_q(\mathrm{PSL}_2)$*. Assume that* $q + q^{-1} \neq 0$*.*

1. $\mathbb{C}$ *has a length* 3 *resolution by relative projective Yetter-Drinfeld modules.*
2. $\mathrm{cd}(A) = 3$*, and if* q *is generic, then* $3 = \mathrm{cd}_{\mathrm{GS}}(A)$*.*
3. *We have*

$$H_b^n(A) \simeq \begin{cases} 0 & \text{if } n \neq 0, 3 \\ \mathbb{C} & \text{if } n = 0, 3 \end{cases}$$

Let $A_s(n)$ be the algebra presented by generators $(u_{ij})_{1 \leq i,j \leq n}$ and relations

$$\sum_k u_{ki} = 1 = \sum_k u_{ik}, \ u_{ik}u_{ij} = \delta_{kj}u_{ij}, \ u_{ki}u_{ji} = \delta_{jk}u_{ji}$$

It has a natural Hopf algebra structure and represents the quantum permutation group S_n^+ (Wang).

Theorem

Let $A = A_s(n)$. Assume that $n \geq 4$.

1. *$\mathbb{C}$ has a length 3 resolution by relative projective Yetter-Drinfeld modules.*
2. $\mathrm{cd}(A) = 3 = \mathrm{cd}_{\mathrm{GS}}(A)$.
3. *We have*

$$H_b^n(A) \simeq \begin{cases} 0 & \text{if } n \neq 0, 3 \\ \mathbb{C} & \text{if } n = 0, 3 \end{cases}$$

As a last example, consider $A_u(n)$, the $*$-algebra presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and relations

$$uu^* = I_n = u^* u, \ \ u^t\overline{u} = I_n = \overline{u}u^t$$

with the obvious Hopf $*$-algebra structure (Wang).

Theorem

We have, for $n \geq 2$

$$\mathrm{cd}(A_u(n)) = 3 = \mathrm{cd}_{\mathrm{GS}}(A_u(n))$$

So the quantum groups O_n^+, S_n^+, U_n^+, that we call free, all have cohomological dimension 3...