FAITHFUL FLATNESS OF HOPF ALGEBRAS OVER COIDEAL SUBALGEBRAS WITH A BIMODULE CONDITIONAL EXPECTATION

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Abstract. We give a direct and self-contained proof that if H is a Hopf algebra and $A \subset H$ is a right coideal subalgebra such A is a direct summand in H as an A-bimodule, then H is faithfully flat as a left and right A-module.

1. Introduction

The aim of this note is to give a direct and self-contained proof of the following result.

Theorem 1.1. *Let H be a Hopf algebra and let $A \subset H$ be a right coideal subalgebra. If A is a direct summand in H as an A-bimodule, then H is faithfully flat as a left and right A-module.*

The original author's interest for Theorem 1.1 came from A. Chirvasitu's result [3, Theorem 2.1] on the faithful flatness of a cosemisimple Hopf algebras over its Hopf subalgebras. Indeed, Chirvasitu's proof is divided in two steps: he first proves the crucial fact that a Hopf subalgebra $A \subset H$ of a cosemisimple Hopf algebra H is a direct summand of H as an A-bimodule, and then concludes using [3, Proposition 1.4, Proposition 1.6], results that are obtained by a discussion involving an important number of external references [2, 4, 6, 7] and various technologies. It is hoped that the present direct and self-contained proof of Theorem 1.1 will provide an easier access to the proof of Chirvasitu's Theorem.

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Notations and conventions. We work over a fixed field k, and assume that the reader is familiar with the theory of Hopf algebras as e.g. in [5]. If H is a Hopf algebra, as usual, Δ, ε and S stand respectively for the comultiplication, counit and antipode of H. We use Sweedler's notations in the standard way.

The category of left A-modules over an algebra is denoted ${}_A\mathcal{M}$, the category of left C-comodules over a coalgebra is denoted ${}^C\mathcal{M}$, etc...

As usual, we say that a right A-module M is flat if the functor $M \otimes_A - : {}_A\mathcal{M} \to {}_k\mathcal{M}$ is exact, which amounts to say map $M \otimes_A -$ preserves injective maps (monomorphisms), and that M is faithfully flat if it is flat and $M \otimes_A -$ creates exact sequences as well. Left (faithfully) flat A-modules are defined similarly. We also say that an algebra extension $A \subset B$ is right or left (faithfully) flat is B is (faithfully) flat as a right or left A-module.

If $A \subset B$ is an algebra extension, then A is direct summand in B as a right A-module if and only there exists a right A-linear map $E : B \to A$ such that $E_{|A} = \mathrm{id}_A$, and we call such a map E is a right conditional expectation for the extension $A \subset B$. The notion of left conditional expectation is defined similarly, and a bimodule conditional expectation is an A-bimodule map $E : B \to A$ such that $E_{|A} = \mathrm{id}_A$.

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2. Proof Of Theorem 1.1

2.1. **Preliminary set-up.** We begin by fixing a number of notation and constructions, which will run throughout the section. All this material can be found in [7].

Let H be a Hopf algebra and let $A \subset H$ be a right coideal subalgebra, which means that A is subalgebra of H such that $\Delta(A) \subset A \otimes H$. Let $A^+ = \mathrm{Ker}(\varepsilon) \cap A$ and consider HA^+, the left ideal of H generated by A^+. It is an immediate verification that HA^+ is a coideal in H ($\Delta(HA^+) \subset HA^+ \otimes H + H \otimes HA^+$ and $\varepsilon(HA^+) = 0$), so we can form the quotient coalgebra $C = H/HA^+$ together with the canonical surjection : $\pi : H \to C = H/HA^+$. The coalgebra C has as well a left H-module structure induced by π, so that C is a left H-module coalgebra. We therefore consider the category of (relative) Hopf modules ${}_H^C\mathcal{M}$, whose objects are the left H-modules and left C-comodules X such that the coaction $\alpha_X : X \to C \otimes X$ is left H-linear, i.e. in Sweedler notation, we have for any $h \in H$ and $x \in X$,

$$(h.x)_{(-1)} \otimes (h.x)_{(0)} = h_{(1)}.x_{(-1)} \otimes h_{(2)}.x_{(0)}$$

For a left A-module M, the induced H-module $H \otimes_A M$ has a left C-comodule structure given by $(h \otimes_A m)_{(-1)} \otimes (h \otimes_A m)_{(0)} = \pi(h_{(1)}) \otimes h_{(2)} \otimes_A m$ making it into an object of ${}_H^C\mathcal{M}$. This defines the induction functor

$$\begin{aligned} L = H \otimes_A - : {}_A\mathcal{M} &\longrightarrow {}_H^C\mathcal{M} \\ M &\longmapsto H \otimes_A M \end{aligned}$$

For an object X in ${}_H^C\mathcal{M}$, let

$${}^{\mathrm{co}C}X = \{x \in X \mid x_{(-1)} \otimes x_{(0)} = \pi(1) \otimes x\}$$

It is immediate to check that ${}^{\mathrm{co}C}X \subset X$ is a sub-A-module and this defines a functor

$$\begin{aligned} R = {}^{\mathrm{co}C}(-) : {}_H^C\mathcal{M} &\longrightarrow {}_A\mathcal{M} \\ X &\longmapsto {}^{\mathrm{co}C}X \end{aligned}$$

which is right adjoint to L. We therefore have a pair of adjoint functors

$$(L, R) : {}_A\mathcal{M} \rightleftarrows {}_H^C\mathcal{M} \tag{2.1}$$

whose respective unit and counit are given by

$$\begin{aligned} \eta_M : M &\longrightarrow {}^{\mathrm{co}C}(H \otimes_A M) & \mu_X : H \otimes_A {}^{\mathrm{co}C}X &\longrightarrow X \\ m &\longmapsto 1_H \otimes_A m & h \otimes x &\longmapsto h.x \end{aligned} \tag{2.2}$$

We have now the necessary material to state the following result, which is [3, Proposition 1.6].

Theorem 2.1. *Let H be a Hopf algebra, let $A \subset H$ be a right coideal subalgebra and let $C = H/HA^+$ be the corresponding quotient coalgebra. The following assertions are equivalent.*

1. *The induction functor ${}_A\mathcal{M} \longrightarrow {}_H^C\mathcal{M}$ is an equivalence of categories.*
2. *The extension $A \subset H$ is right faithfully flat.*
3. *The above unit and counit morphisms (2.2) are isomorphisms.*

It is immediate that (1) $\Rightarrow$ (2) since an equivalence of categories is a faithfully exact functor and the exact sequences in ${}_A\mathcal{M}$ and ${}_H^C\mathcal{M}$ are precisely those that are exact in ${}_k\mathcal{M}$. It is clear that (3) $\Rightarrow$ (1). The arguments we develop to prove Theorem 1.1 will provide as well a proof of (2)$\Rightarrow$(3).

2.2. **The canonical isomorphisms.** We will use some "canonical" isomorphisms, that we construct in this subsection.

For a left H-module X, endow $C \otimes X$ with the tensor product left H-module structure and with the left C-comodule structure provided by the comultiplication of C. In this way $C \otimes X$ becomes an object of ${}^C_H\mathcal{M}$ (in fact $C \otimes X$ is the image of X by the right adjoint to the forgetful functor ${}^C_H\mathcal{M} \to {}_H\mathcal{M}$).

Proposition 2.2. *Let $A \subset H$ be a right coideal subalgebra and let X be left H-module. The canonical map*

$$\begin{aligned} \kappa_X : H \otimes_A X &\longrightarrow C \otimes X \\ h \otimes_A x &\longmapsto \pi(h_{(1)}) \otimes h_{(2)}.x \end{aligned}$$

is an isomorphism in the category ${}^C_H\mathcal{M}$.

Proof. It is a direct verification that κ_X is a a morphism in ${}^C_H\mathcal{M}$, and that

$$\begin{aligned} C \otimes X &\longrightarrow H \otimes_A X \\ \pi(h) \otimes x &\longmapsto h_{(1)} \otimes_A S(h_{(2)}).x \end{aligned}$$

is the inverse isomorphism. □

2.3. **The unit of the adjunction.** We first analyse the unit of our adjunction, starting with a general observation.

Proposition 2.3. *Let $A \subset B$ be an algebra extension, let M be a left A-module M, and consider the map*

$$\begin{aligned} \iota_M : M &\longrightarrow \left\{ \sum_i x_i \otimes_A m_i \in B \otimes_A M \;\middle|\; \sum_i x_i \otimes_A 1_B \otimes_A m_i = \sum_i 1_B \otimes_A x_i \otimes_A m_i \right\} \\ m &\longmapsto 1_B \otimes_A m \end{aligned}$$

The ι_M is an isomorphism if one of the following conditions holds:

(1) $A \subset B$ is right faithfully flat;

(2) A is a direct summand in B as a right A-module.

Proof. It is a well-known result (see e.g. the second theorem of Section 13.1 in [8]) that ι_M is an isomorphism if B is faithfully flat as a right A-module, that we do not reproduce here.

Let $E : B \to A$ be a right conditional expectation. The right A-linearity of E enables us to define, for any left A-module M, the map

$$\begin{aligned} E_M : B \otimes_A M &\to M \\ x \otimes_A m &\mapsto E(x).m \end{aligned}$$

For simplicity denote $X(M)$ the space on the right. Let us check that $E_{M|X(M)} : X(M) \to M$ is an inverse isomorphism to ι_M. In one direction it is clear that $E_M \circ \iota_M = \mathrm{id}_M$. To prove that $\iota_M \circ E_{M|X(M)} = \mathrm{id}_{X(M)}$, similarly to before, notice that the right A-linearity of E enables us to define the map

$$\begin{aligned} E'_M : B \otimes_A B \otimes_A M &\to B \otimes_A M \\ x \otimes_A y \otimes_A m &\mapsto E(x)y \otimes_A m \end{aligned}$$

For $\sum_i x_i \otimes_A m_i \in X(M)$, we have

$$\sum_i 1_B \otimes_A x_i \otimes_A m_i = \sum_i x_i \otimes_A 1_B \otimes_A m_i$$

and applying E'_M yields

$$\sum_i x_i \otimes_A m_i = \sum_i E(x_i) \otimes_A m_i = 1_B \otimes_A \left(\sum_i E(x_i).m_i\right) = \iota_M \circ E_M \left(\sum_i x_i \otimes_A m_i\right)$$

which concludes the proof. □

Proposition 2.4. *Let $A \subset H$ be a right coideal subalgebra. Assume that $A \subset H$ is right faithfully flat or that A is a direct summand in H as a right A-module. Then for any left A-module M, the unit map*

$$\eta_M : M \longrightarrow {}^{\mathrm{co}C}(H \otimes_A M)$$

is an isomorphism.

Proof. We consider the canonical isomorphism

$$\begin{aligned} \kappa'_M = \kappa_{H \otimes_A M} : H \otimes_A (H \otimes_A M) &\longrightarrow C \otimes (H \otimes_A M) \\ h \otimes_A h' \otimes_A m &\longmapsto \pi(h_{(1)}) \otimes h_{(2)} h' \otimes_A m \end{aligned}$$

from Proposition 2.2. For $\sum_i h_i \otimes_A m_i \in {}^{\mathrm{co}C}(H \otimes_A M)$, we have

$$\kappa'_M \left(\sum_i h_i \otimes_A 1_H \otimes_A m_i\right) = \sum_i \pi(1) \otimes h_i \otimes_A m_i = \kappa'_M \left(\sum_i 1_H \otimes_A h_i \otimes_A m_i\right)$$

and the injectivity of κ'_M gives

$$\sum_i h_i \otimes_A 1_H \otimes_A m_i = \sum_i 1_H \otimes_A h_i \otimes_A m_i$$

Our assumption then ensures, by Proposition 2.3, the existence of a unique $m \in M$ such that $\sum_i h_i \otimes_A m_i = 1_H \otimes_A m$. This therefore defines a map ${}^{\mathrm{co}C}(H \otimes_A M) \to M$, which is clearly an inverse to η_M. □

Remark 2.5. If A is a direct summand in H as a right A-module, Proposition 2.4 ensures in particular that ${}^{\mathrm{co}C}H = A$. Hence in view of Proposition 2.2, we see that $A \subset H$ is coalgebra Galois extension over C. Once this is noticed, the shortest way to obtain a proof of Theorem 1.1 is certainly to invoke [1, Proposition 4.4]. I thank B. Mesablishvili for pointing out coalgebra Galois extension in this context.

2.4. **The counit of the adjunction.** We now analyse the counit of our adjunction (L, R). We begin with a lemma, in which we use the following notation: if X is an object of ${}^C_H\mathcal{M}$, we denote $i_X : {}^{\mathrm{co}C}X \to X$ the natural inclusion map.

Lemma 2.6. *Let $A \subset H$ be a right coideal subalgebra and let X be an object of ${}^C_H\mathcal{M}$. Assume that A is a direct summand in H as a right A-module, and let $E : H \to A$ be a right conditional expectation. Then the map*

$$p_X = (E \otimes_A \mathrm{id}_X) \circ \kappa_X^{-1} \circ \alpha_X : X \to X$$

is a projection of X onto ${}^{\mathrm{co}C}X$. If moreover E is an A-bimodule conditional expectation, then $p_X : X \to {}^{\mathrm{co}C}X$ is A-linear, and hence the map $\mathrm{id}_H \otimes_A i_X : H \otimes_A {}^{\mathrm{co}C}X \to H \otimes_A X$ is injective.

Proof. The above map p_X is well-defined since E is right A-linear, is an A-bimodule map as soon as E is, and for $x \in {}^{\mathrm{co}C}X$, it is clear that $p_X(x) = x$, since $E(1_H) = 1_A$. Thus one just has to check that p_X has values into ${}^{\mathrm{co}C}X$, which follows from the commutativity of the following diagram:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\alpha_X} & C \otimes X & \xrightarrow{\kappa_X^{-1}} & H \otimes_A X & \xrightarrow{E \otimes_A \mathrm{id}_X} & X \\
\downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \alpha_X} & & \downarrow{\scriptstyle \mathrm{id}_H \otimes_A \alpha_X} & & \downarrow{\scriptstyle \alpha_X} \\
C \otimes X & \xrightarrow{\Delta_C \otimes \mathrm{id}_X} & C \otimes (C \otimes X) & \xrightarrow{\kappa_{C \otimes X}^{-1}} & H \otimes_A (C \otimes X) & \xrightarrow{E \otimes_A \otimes \mathrm{id}_{C \otimes X}} & C \otimes X \\
& \searrow{\scriptstyle \kappa_X^{-1}} & & \nearrow{\scriptstyle \nu_X} & & & \\
& & H \otimes_A X & & & &
\end{array}
$$

where $\nu_X : H \otimes_A X : H \otimes A(C \otimes X)$ is defined by $\nu_X(h \otimes_A x) = h \otimes_A \pi(1) \otimes x$.

If E is an A-bimodule map, then p_X is left A-linear and therefore $\mathrm{id}_H \otimes_A p_X$ provides a retraction to $\mathrm{id}_H \otimes_A i_X$, which gives the last statement. $\square$

Proposition 2.7. *Let $A \subset H$ be a right coideal subalgebra and let X be an object of ${}^C_H\mathcal{M}$. Consider the following assertions:*

(a) $\mathrm{id}_H \otimes_A i_X : H \otimes_A {}^{\mathrm{co}C}X \to H \otimes_A X$ *is injective;*
(b) $\mu_X : H \otimes_A {}^{\mathrm{co}C}X \to X$ *is injective;*
(c) $\mu_X : H \otimes_A {}^{\mathrm{co}C}X \to X$ *is surjective.*

Then we have $(a) \iff (b) \implies (c)$. These assertions hold true if one of the following conditions is satisfied:

(1) H is flat as a right A-module;
(2) H is a direct summand in B as an A-bimodule.

Proof. Consider the map $\delta : X \to C \otimes X$ defined by $\delta(x) = x_{(-1)} \otimes x_{(0)} - \pi(1) \otimes x$. This map is A-linear and the sequence of A-modules $0 \to {}^{\mathrm{co}C}X \xrightarrow{i_X} X \xrightarrow{\delta} C \otimes X$ is exact. Applying $H \otimes_A -$ yields the sequence

$$H \otimes_A {}^{\mathrm{co}C}X \xrightarrow{\mathrm{id}_H \otimes_A i_X} H \otimes_A X \xrightarrow{\mathrm{id}_H \otimes_A \delta} H \otimes_A (C \otimes X)$$

that fits in the commutative diagram

$$
\begin{array}{ccccccc}
 & & H \otimes_A {}^{\mathrm{co}C}X & \xrightarrow{\mathrm{id}_H \otimes_A i} & H \otimes_A X & \xrightarrow{\mathrm{id}_H \otimes_A \delta} & H \otimes_A (C \otimes X) \\
 & & \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \kappa_X} & & \downarrow{\scriptstyle \kappa_{C \otimes X}} \\
0 & \longrightarrow & X & \xrightarrow{\alpha_X} & C \otimes X & \xrightarrow{\nabla} & C \otimes (C \otimes X)
\end{array}
$$

where $\nabla : C \otimes X \to C \otimes (C \otimes X)$ is defined by

$$\nabla(\pi(h) \otimes x) = \pi(h) \otimes x_{(-1)} \otimes x_{(0)} - \pi(h_{(1)}) \otimes \pi(h_{(2)}) \otimes x$$

Since $\kappa_X^{-1} \circ \alpha_X$ is injective, we get $(a) \iff (b)$.

Assume that (a) holds. Then the upper sequence in the above diagram is exact (while the lower row is exact by construction), and it is a simple diagram chasing to check that μ_X is surjective.

If (1) holds, then (a) holds by the definition of flatness, and if (2) holds, Lemma 2.6 ensures that (a) holds. $\square$

The proof of right faithful flatness in Theorem 1.1 is now immediate: if A is a direct summand in H as an A-bimodule, Proposition 2.4 ensures that the unit of the adjunction (L, R) is an isomorphism, and Proposition 2.7 ensures that the counit is an isomorphism as well, so $B \subset H$ is right faithfully flat by (3)⇒(2) in Theorem 2.1.

Under the assumption that A is a direct summand in H as an A-bimodule, left faithful flatness is shown similarly by considering the right H-module quotient coalgebra $D = H/A^+H$, and the category $\mathcal{M}_H^D$, we do not write the details.

Notice as well that Proposition 2.4 and Proposition 2.7 combined together immediately show (2)⇒(3) in Theorem 2.1.

3. Concluding remark

It is unclear to us whether the the assumption "A is a direct summand in H as an A-bimodule" in Theorem 1.1 can be weakened to "A is direct summand as a right H-module" to conclude that H is right faithfully flat over A. In this concluding section we summarize what is known from the previous section.

Proposition 3.1. *Let $A \subset H$ be a right coideal subalgebra. Assume that A is a direct summand in H as a right A-module. Then the following assertions are equivalent:*

(1) H is faithfully flat as a right A-module;
(2) H is flat as a right A-module;
(3) For any object X of ${}_H^C\mathcal{M}$, the map $\mathrm{id}_H \otimes_A i_X : H \otimes_A {}^{\mathrm{co}C}X \to H \otimes_A X$ is injective;
(4) For any object X of ${}_H^C\mathcal{M}$, the map $\mu_X : H \otimes_H {}^{\mathrm{co}C}X \to X$ is surjective.

Proof. By definition (1)⇒(2) and (2)⇒(3), while (3)⇒ (4) follows from Proposition 2.7. Assume that (4) holds. Since A is a direct summand in B as a right A-module, by Proposition 2.4 we are in the situation of a pair of adjoint functors (L, R) whose unit is an isomorphism and counit is an epimorphism: it is then easy to check that R faithful, and that the counit is a monomorphism as well, so that L and R are inverse equivalences (since we are dealing with categories in which morphisms that are both monomorphisms and epimorphisms are isomorphisms). Hence H is faithfully flat as a right A-module. □

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