FAITHFUL FLATNESS OF HOPF ALGEBRAS OVER COIDEAL SUBALGEBRAS WITH A CONDITIONAL EXPECTATION

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ABSTRACT. Let H be a Hopf algebra and let $A \subset H$ be a right coideal subalgebra. We show that if A is a direct summand in H as a right A-module, then H is faithfully flat as a right A-module.

1. Introduction

Faithful flatness is an important algebraic concept arising from algebraic geometry and closely related to the question of forming "good" quotients. It takes a key role in the theory of group schemes and their noncommutative generalizations, Hopf algebras, and especially in questions related to exact sequences [1].

The question whether a Hopf algebra is faithfully flat over its Hopf subalgebras was raised in Montgomery's book [11]. Schauenburg [15] has shown that the answer is negative in general, but the examples considered there are due to a somewhat pathological behavior of the antipodes, and it is expected that faithful flatness over Hopf subalgebras should hold in the most reasonable situations. Here are some highlights of the known positive results ($A \subset H$ is a Hopf subalgebra in what follows):

(1) First there is the classical result that if H is commutative then it is faithfully flat over its Hopf subalgebras. This is precisely one of the ingredients that ensures that there is a good quotient theory for affine group schemes, see [8, 19, 22].
(2) The situation when H is commutative has been generalized to the case when only A is commutative, by Arkhipov-Gaitsgory [2], and to the case when H is Noetherian, residually finite-dimensional and has a polynomial identity, by Skryabin [18].
(3) Pointed Hopf algebras are free over their Hopf subalgebras: this was shown by Radford [14].
(4) The celebrated Nichols-Zoeller theorem [12] ensures that finite-dimensional Hopf algebras are free over their Hopf subalgebras.
(5) Chirvasitu [6] has shown that if H is cosemisimple, then it is faithfully flat over its Hopf subalgebras.

The case when $A \subset H$ is only a coideal subalgebra is also of high interest in view of quotient theory, but here the situation is not that we can reasonably expect that faithful flatness holds, since even in the commutative case it is easy to find natural coideal subalgebras over which the Hopf algebra is not faithfully flat (for example $k[x, y] \subset \mathcal{O}(\mathrm{SL}_2(k))$. There are however a number of positive results, possibly by relaxing faithful flatness to flatness, of which we also list a selection.

(1) Masuoka-Wigner [9] have shown that commutative Hopf algebras are flat over their coideal subalgebras, and this was generalized by Skryabin [18] the case when H is Noetherian, residually finite-dimensional and has a polynomial identity.

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(2) Skryabin [17] has shown that finite-dimensional Hopf algebras are free over their coideal subalgebras.
(3) Compact Hopf algebras are faithfully flat over their coideal $*$-subalgebras: this was shown by Chirvasitu [7].

The primary motivation for this work was the idea that it would be useful to write a self-contained proof of the above mentioned result [6, Theorem 2.1] of Chirvasitu on the faithful flatness of a cosemisimple Hopf algebra over its Hopf subalgebras. Indeed, Chirvasitu's proof is divided in two steps: he first proves the crucial fact that a Hopf subalgebra $A \subset H$ of a cosemisimple H is a direct summand of H as an A-bimodule, and then concludes using [6, Proposition 1.4, Proposition 1.6], results that are obtained by a discussion involving an important number of external references [5, 10, 13, 20]. As a result of the writing of the self-contained proof, we obtain the following new faithful flatness result, in the setting of coideal subalgebras.

Theorem 1.1. *Let H be a Hopf algebra and let $A \subset H$ be a right coideal subalgebra. If A is a direct summand in H as a right A-module, then H is faithfully flat as a right A-module.*

The condition that A is a direct summand in H as a right A-module is equivalent to the existence of a right A-linear map $E : H \to A$ such that $E_{|A} = \mathrm{id}_A$. Such a map E is what we call a right conditional expectation for the extension $A \subset H$.

The work of Takeuchi [20] revealed that the question of faithful flatness is intimately related to the structure of certain categories of relative Hopf modules, and the proof of Theorem 1.1 is based on such a structure theorem, Theorem 3.1 in Section 3. This result is a generalization of [6, Proposition 1.6], involving a notion that we call faithful flatishness, and which is in general a technical consequence of faithful flatness. The key point is that faithful flatishness is an easy consequence of the existence of a conditional expectation, and is sufficient to prove the structure theorem of relative Hopf modules. Several arguments in the proof of Theorem 3.1 are close to classical arguments that one finds here and there in the literature, and it is fair to say that the proof of [16, Theorem 3.7] was particularly inspiring.

The paper is organized as follows: in Section 2 we introduce the notion of a faithfully flatish extension, while in Section 3, after having recalled the adequate framework of relative Hopf modules, we prove Theorem 3.1, from which Theorem 1.1 follows. The final Section 4 discusses some left/right variations and consequences of Theorem 1.1, and an illustration on an example.

Notations and conventions. We work over a fixed field k, and assume that the reader is familiar with the theory of Hopf algebras as e.g. in [11]. If H is a Hopf algebra, as usual, Δ, ε and S stand respectively for the comultiplication, counit and antipode of H. We use Sweedler's notations in the standard way. The category of left A-modules over an algebra is denoted ${}_A\mathcal{M}$, the category of left C-comodules over a coalgebra is denoted ${}^C\mathcal{M}$, etc...

2. Faithfully flat and faithfully flatish extensions

As usual, we say that a right A-module M is flat if the functor $M \otimes_A - : {}_A\mathcal{M} \to {}_k\mathcal{M}$ is exact, which amounts to say map $M \otimes_A -$ preserves injective maps (monomorphisms), and that M is faithfully flat if it is flat and $M \otimes_A -$ creates exact sequences as well. We also say that an algebra extension $A \subset B$ is right (faithfully) flat is B is (faithfully) flat as a right A-module.

We introduce a technical notion, which is in general a consequence of faithful flatness, but which is sufficient in many applications.

Definition 2.1. An algebra extension $A \subset B$ is said to be right faithfully flatish if for any left A-module M, the map

$$\begin{aligned} \iota_M : M &\longrightarrow \left\{ \sum_i x_i \otimes_A m_i \in B \otimes_A M \;\middle|\; \sum_i x_i \otimes_A 1_B \otimes_A m_i = \sum_i 1_B \otimes_A x_i \otimes_A m_i \right\} \\ m &\longmapsto 1_B \otimes_A m \end{aligned}$$

is an isomorphism.

It is a well-known property that faithfully flat extensions are faithfully flatish, see e.g. the second theorem of Section 13.1 in [22]. The following result ensures that faithful flatishness holds in the context of Theorem 1.1.

Proposition 2.2. *Let $A \subset B$ be an algebra extension, and assume that A is a direct summand in B as a right A-module. Then the extension $A \subset B$ is right faithfully flatish.*

Proof. Let $E : B \to A$ be a right conditional expectation. The right A-linearity of E enables us to define, for any left A-module M, the map

$$\begin{aligned} E_M : B \otimes_A M &\to M \\ x \otimes_A m &\mapsto E(x).m \end{aligned}$$

Denote by $X(M)$ the A-module on the right in Definition 2.1. Let us check that $E_{M|X(M)} : X(M) \to M$ is a reciprocal isomorphism to ι_M in Definition 2.1. In one direction it is clear that $E_M \circ \iota_M = \mathrm{id}_M$. To prove that $\iota_M \circ E_{M|X(M)} = \mathrm{id}_{X(M)}$, similarly to before, notice that the right A-linearity of E enables us to define the map

$$\begin{aligned} E'_M : B \otimes_A B \otimes_A M &\to B \otimes_A M \\ x \otimes_A y \otimes_A m &\mapsto E(x)y \otimes_A m \end{aligned}$$

For $\sum_i x_i \otimes_A m_i \in X(M)$, we have

$$\sum_i 1_B \otimes_A x_i \otimes_A m_i = \sum_i x_i \otimes_A 1_B \otimes_A m_i$$

and applying E'_M yields

$$\sum_i x_i \otimes_A m_i = \sum_i E(x_i) \otimes_A m_i = 1_B \otimes_A \left(\sum_i E(x_i).m_i \right) = \iota_M \circ E_M \left(\sum_i x_i \otimes_A m_i \right)$$

which concludes the proof. $\square$

3. Relative Hopf modules and faithful flatishness

In this section we show that for a right coideal subalgebra $A \subset H$, faithful flatishness is equivalent to faithful flatness, which will prove Theorem 1.1. This is done in the context of a structure theorem for a certain category of Hopf modules introduced by Takeuchi [20].

3.1. **Preliminary set-up.** We begin by fixing a number of notation and constructions, which will run throughout the section. All this material can be found in [20].

Let H be a Hopf algebra and let $A \subset H$ be a right coideal subalgebra, which means that A is subalgebra of H such that $\Delta(A) \subset A \otimes H$. Let $A^+ = \mathrm{Ker}(\varepsilon) \cap A$ and consider HA^+, the left ideal of H generated by A^+. It is an immediate verification that HA^+ is a coideal in H ($\Delta(HA^+) \subset HA^+ \otimes H + H \otimes HA^+$ and $\varepsilon(HA^+) = 0$), so we can form the quotient coalgebra $C = H/HA^+$ together with the canonical surjection : $\pi : H \to C = H/HA^+$. The coalgebra C has as well a left H-module structure induced by π, so that C is a left H-module coalgebra. We therefore consider the category of (relative) Hopf modules ${}_H^C\mathcal{M}$, whose objects are the left H-modules and left C-comodules X such that the coaction $\alpha_X : X \to C \otimes X$ is left H-linear, i.e. in Sweedler notation, we have for any $h \in H$ and $x \in X$,

$$(h.x)_{(-1)} \otimes (h.x)_{(0)} = h_{(1)}.x_{(-1)} \otimes h_{(2)}.x_{(0)}$$

For a left A-module M, the induced H-module $H \otimes_A M$ has a left C-comodule structure given by $(h \otimes_A m)_{(-1} \otimes (h \otimes_A m)_{(0)} = \pi(h_{(1)}) \otimes h_{(2)} \otimes_A m$ making it into an object of ${}_H^C\mathcal{M}$. This defines the induction functor

$$\begin{aligned} L = H \otimes_A - : {}_A\mathcal{M} &\longrightarrow {}_H^C\mathcal{M} \\ M &\longmapsto H \otimes_A M \end{aligned}$$

For an object X in ${}_H^C\mathcal{M}$, let

$${}^{\mathrm{co}C}X = \{x \in X \mid x_{(-1)} \otimes x_{(0)} = \pi(1) \otimes x\}$$

It is immediate to check that ${}^{\mathrm{co}C}X \subset X$ is a sub-A-module and this defines a functor

$$\begin{aligned} R = {}^{\mathrm{co}C}(-) : {}_H^C\mathcal{M} &\longrightarrow {}_A\mathcal{M} \\ X &\longmapsto {}^{\mathrm{co}C}X \end{aligned}$$

which is right adjoint to L. We therefore have a pair of adjoint functors

$$(L, R) : {}_A\mathcal{M} \longrightarrow {}_H^C\mathcal{M} \tag{3.1}$$

whose respective unit and counit are given by

$$\begin{aligned} \eta_M : M &\longrightarrow {}^{\mathrm{co}C}(H \otimes_A M) \\ m &\longmapsto 1_H \otimes_A m \end{aligned} \qquad \begin{aligned} \mu_X : H \otimes_A {}^{\mathrm{co}C}X &\longrightarrow X \\ h \otimes x &\longmapsto h.x \end{aligned}$$

3.2. **Statement of the main result.** We have now the necessary material to state the main result of the section, as follows.

Theorem 3.1. *Let H be a Hopf algebra, let $A \subset H$ be a right coideal subalgebra and let $C = H/HA^+$ be the corresponding quotient coalgebra. The following assertions are equivalent.*

1. *The induction functor ${}_A\mathcal{M} \longrightarrow {}_H^C\mathcal{M}$ is an equivalence of categories.*
2. *The extension $A \subset H$ is right faithfully flat.*
3. *The extension $A \subset H$ is right faithfully flatish.*

Theorem 1.1 is an immediate consequence of the combination of Theorem 3.1 and Proposition 2.2.

It is immediate that (1) ⇒ (2) since an equivalence of categories is a faithfully exact functor and the exact sequences in ${}_A\mathcal{M}$ and ${}_H^C\mathcal{M}$ are precisely those that are exact in ${}_k\mathcal{M}$, and it has already be been mentioned that (2) ⇒ (3). The rest of this section is devoted to the proof of (3) ⇒ (1), which will consist of showing that the unit and counit of the pair of adjoint functors (L, R) in (3.1) are isomorphims, and will be done in several steps.

3.3. **The canonical isomorphisms.** The proof of Theorem 3.1 will require a number of "canonical" isomorphisms, that we construct in this subsection.

For a left H-module X, endow $C \otimes X$ with the tensor product left H-module structure and with the left C-comodule structure provided by the comultiplication of C. In this way $C \otimes X$ becomes an object of ${}^C_H\mathcal{M}$ (in fact $C \otimes X$ is the image of X by the right adjoint to the forgetful functor ${}^C_H\mathcal{M} \to {}_H\mathcal{M}$).

Our first canonical isomorphism is as follows.

Proposition 3.2. *Let X be left H-module. The canonical map*

$$\begin{aligned} \kappa_X : H \otimes_A X &\longrightarrow C \otimes X \\ h \otimes_A x &\longmapsto \pi(h_{(1)}) \otimes h_{(2)}.x \end{aligned}$$

is an isomorphism in the category ${}^C_H\mathcal{M}$.

Proof. It is a direct verification that κ_X is a a morphism in ${}^C_H\mathcal{M}$, and that

$$\begin{aligned} C \otimes X &\longrightarrow H \otimes_A X \\ \pi(h) \otimes x &\longmapsto h_{(1)} \otimes_A S(h_{(2)}).x \end{aligned}$$

is the inverse isomorphism. □

We will need also a variation on the above canonical isomorphisms, for which we introduce some more notation. For an object X in ${}^C_H\mathcal{M}$, we consider the following maps:

$$\begin{aligned} \delta = \delta_X : X &\longrightarrow C \otimes X \\ x &\longmapsto x_{(-1)} \otimes x_{(0)} - \pi(1) \otimes x \end{aligned}$$

$$\begin{aligned} \nabla = \nabla_X : C \otimes X &\longrightarrow C \otimes (C \otimes X) \\ \pi(h) \otimes x &\longmapsto \pi(h) \otimes x_{(-1)} \otimes x_{(0)} - \pi(h_{(1)}) \otimes \pi(h_{(2)}) \otimes x \end{aligned}$$

It is immediate that δ is left A-linear, while ∇ is a morphism in ${}^C_H\mathcal{M}$.

Proposition 3.3. *Let X be an object in ${}^C_H\mathcal{M}$. We have an isomorphism*

$$\begin{aligned} \widetilde{\kappa}_X : H \otimes_A \delta(X) &\longrightarrow \nabla(C \otimes X) \\ h \otimes_A \delta(x) &\longmapsto \nabla\left(\pi(h_{(1)}) \otimes h_{(2)}.x\right) \end{aligned}$$

making the following diagram commute

$$\begin{tikzcd} H \otimes_A X \arrow[r, "\mathrm{id}_H \otimes_A \delta"] \arrow[d, "\kappa_X"] & H \otimes_A \delta(X) \arrow[d, "\widetilde{\kappa}_X"] \\ C \otimes X \arrow[r, "\nabla"] & \nabla(C \otimes X) \end{tikzcd}$$

Proof. Consider the map $\kappa_X^0 : H \otimes_A X \to C \otimes (C \otimes X)$ given by the composition

$$H \otimes_A X \xrightarrow{\mathrm{id}_H \otimes_A \delta} H \otimes_A (C \otimes X) \xrightarrow{\kappa_{C \otimes X}} C \otimes (C \otimes X)$$

For $h, h' \in H$ and $x \in X$, we have $\kappa_{C\otimes X}(h \otimes_A \pi(h') \otimes x) = \pi(h_{(1)}) \otimes \pi(h_{(2)}h') \otimes h_{(3)}.x$, hence

$$\begin{aligned}
\kappa^0_X(h \otimes_A x) &= \kappa_{C\otimes X}(h \otimes_A x_{(-1)} \otimes x_{(0)} - h \otimes_A \pi(1) \otimes x) \\
&= \pi(h_{(1)}) \otimes h_{(2)}.x_{(-1)} \otimes h_{(3)}.x_{(0)} - \pi(h_{(1)}) \otimes \pi(h_{(2)}) \otimes h_{(3)}.x \\
&= \nabla\left(\pi(h_{(1)}) \otimes h_{(2)}.x\right)
\end{aligned}$$

This means that the following diagram commutes

$$\begin{tikzcd}
H \otimes_A X \arrow[r, "\mathrm{id}_H \otimes_A \delta"] \arrow[d, "\kappa_X"] \arrow[dr, "\kappa^0_X"] & H \otimes_A (C \otimes X) \arrow[d, "\kappa_{C\otimes X}"] \\
C \otimes X \arrow[r, "\nabla"] & C \otimes (C \otimes X)
\end{tikzcd}$$

For $x \in \mathrm{Ker}(\delta) = {}^{\mathrm{co}C}X$, we then have $\kappa^0_X(h \otimes_A x) = 0$, hence, since $\delta(X) \simeq X/\mathrm{Ker}(\delta)$, we get that κ^0_X induces the announced map $\widetilde{\kappa}_X : H \otimes_A \delta(X) \to \nabla(C \otimes X)$.

To construct the inverse map, consider the composition

$$(\mathrm{id}_H \otimes_A \delta) \circ \kappa_X^{-1} : C \otimes X \to H \otimes_A \delta(X)$$

In view of the previous commutative diagram, this map coincides with $\kappa^{-1}_{C\otimes X} \circ \nabla$, hence vanishes on $\mathrm{Ker}(\nabla)$, and thus induces

$$\begin{aligned}
\nabla(C \otimes X) &\longrightarrow H \otimes_A \delta(X) \\
\nabla(\pi(h) \otimes x) &\longmapsto h_{(1)} \otimes \delta(S(h_{(2)}).x)
\end{aligned}$$

which is clearly an inverse isomorphism to $\widetilde{\kappa}_X$. □

3.4. **The counit of the adjunction.** We now analyse the counit of our adjunction (L, R).

Proposition 3.4. *Let X be an object in ${}^C_H\mathcal{M}$. The counit map $\mu_X : H \otimes_A {}^{\mathrm{co}C}X \to X$ is surjective, and hence the functor $R = {}^{\mathrm{co}C}(-) : {}^C_H\mathcal{M} \to {}_A\mathcal{M}$ is faithful. If moreover H is flat as a right A-module, then μ_X is an isomorphism.*

Proof. Starting with the exact sequence of A-modules $0 \to {}^{\mathrm{co}C}X \xrightarrow{i} X \xrightarrow{\delta} \delta(X) \to 0$ and applying $H \otimes_A -$ yields the exact sequence

$$H \otimes_A {}^{\mathrm{co}C}X \xrightarrow{\mathrm{id}_H \otimes_A i} H \otimes_A X \xrightarrow{\mathrm{id}_H \otimes_A \delta} H \otimes_A \delta(X) \to 0$$

that fits in the commutative diagram

$$\begin{tikzcd}
 & H \otimes_A {}^{\mathrm{co}C}X \arrow[r, "\mathrm{id}_H \otimes_A i"] \arrow[d, "\mu_X"] & H \otimes_A X \arrow[r, "\mathrm{id}_H \otimes_A \delta"] \arrow[d, "\kappa_X"] & H \otimes_A \delta(X) \arrow[r] \arrow[d, "\widetilde{\kappa}_X"] & 0 \\
0 \arrow[r] & X \arrow[r, "\alpha_X"] & C \otimes X \arrow[r, "\nabla"] & \nabla(C \otimes X) \arrow[r] & 0
\end{tikzcd}$$

where κ_X and $\widetilde{\kappa}_X$ are the isomorphisms of Proposition 3.2 and Proposition 3.3 respectively and the bottom row is exact. The surjectivity of μ_X is then obtained from an easy diagram chasing. If $f : X \to Y$ is a morphism in ${}^C_H\mathcal{M}$, the commutativity of the diagram

$$\begin{tikzcd}
H \otimes_A {}^{\mathrm{co}C}X \arrow[r, "\mathrm{id}_H \otimes_A f_|"] \arrow[d, "\mu_X"] & H \otimes_A {}^{\mathrm{co}C}Y \arrow[d, "\mu_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

together with the surjectivity of μ_X ensures that if $f_{|^{\mathrm{co}C}X} = 0$, then $f = 0$, and hence the functor R is faithful.

Assuming moreover that H is flat as a right A-module enables us, as in the proof of [16, Theorem 3.7], to enlarge the previous commutative diagram to the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H \otimes_A {}^{\mathrm{co}C}X \arrow[r, "\mathrm{id}_H \otimes_A i"] \arrow[d, "\mu_X"] & H \otimes_A X \arrow[r, "\mathrm{id}_H \otimes_A \delta"] \arrow[d, "\kappa_X"] & H \otimes_A \delta(X) \arrow[r] \arrow[d, "\widetilde{\kappa}_X"] & 0 \\
0 \arrow[r] & X \arrow[r, "\alpha_X"] & C \otimes X \arrow[r, "\nabla"] & \nabla(C \otimes X) \arrow[r] & 0
\end{tikzcd}$$

and again a diagram chasing gives that μ_X is an isomorphism. $\square$

3.5. **The unit of the adjunction.** The next step is to study the unit of our adjunction.

Proposition 3.5. *Assume that $A \subset H$ is right faithfully flatish. Then for any left A-module M, the unit map*

$$\eta_M : M \longrightarrow {}^{\mathrm{co}C}(H \otimes_A M)$$

is an isomorphism.

Proof. We consider the canonical isomorphism

$$\begin{aligned}
\kappa'_M = \kappa_{H \otimes_A M} : H \otimes_A (H \otimes_A M) &\longrightarrow C \otimes (H \otimes_A M) \\
h \otimes_A h' \otimes_A m &\longmapsto \pi(h_{(1)}) \otimes h_{(2)} h' \otimes_A m
\end{aligned}$$

from Proposition 3.2. For $\sum_i h_i \otimes_A m_i \in {}^{\mathrm{co}C}(H \otimes_A M)$, we have

$$\kappa'_M \left(\sum_i h_i \otimes_A 1_H \otimes_A m_i \right) = \sum_i \pi(1) \otimes_A h_i \otimes_A m_i = \kappa'_M \left(\sum_i 1_H \otimes_A h_i \otimes_A m_i \right)$$

and the injectivity of κ'_M gives

$$\sum_i h_i \otimes_A 1_H \otimes_A m_i = \sum_i 1_H \otimes_A h_i \otimes_A m_i$$

The faithful flatishness assumption then ensures the existence of a unique $m \in M$ such that $\sum_i h_i \otimes_A m_i = 1_H \otimes_A m$. This therefore defines a map ${}^{\mathrm{co}C}(H \otimes_A M) \to M$, which is clearly an inverse to η_M. $\square$

3.6. **Proof of Theorem 3.1.** We need a last elementary lemma:

Lemma 3.6. *Let $(F, G) : \mathcal{C} \to \mathcal{D}$ be a pair of adjoint functors. Assume that the corresponding unit $\eta : 1_{\mathcal{C}} \to GF$ is an isomorphism and that G is faithful. Then F preserves monomorphisms.*

Proof. Let $u : X \to Y$ be a monomorphism in $\mathcal{C}$, and let $f, g : V \to F(X)$ be morphisms in $\mathcal{D}$ such that $F(u) \circ f = F(u) \circ g$. Then $GF(u) \circ G(f) = GF(u) \circ G(g)$, and since $GF(u) \circ \eta_X = \eta_Y \circ u$, we obtain, because η_Y and η_X are isomorphisms and u is a monomorphism, that $G(f) = G(g)$, and we conclude $f = g$ by faithfulness of G. $\square$

We can now finish the proof of (3)⇒(1) in Theorem 3.1. Assume that $A \subset H$ is right faithfully flatish. Proposition 3.5 then ensures that the unit of the adjunction (L, R) is an isomorphism. Moreover the functor R is faithful by Proposition 3.4, and hence L preserves monomorphisms by Lemma 3.6, which precisely means that H is flat as a right A-module. Then the last statement in Proposition 3.4 ensures that the counit of the adjunction (L, R) is an isomorphism as well, so that (L, R) forms a pair of inverse equivalences of categories.

4. Consequences and an example

4.1. **Left/right variations.** Theorem 1.1 has an obvious left version:

Theorem 4.1. *Let H be a Hopf algebra and let $A \subset H$ be a left coideal subalgebra. If A is a direct summand in H as a left A-module, then H is faithfully flat as a left A-module.*

Proof. This follows from Theorem 1.1, applied to the right coideal subalgebra $A^{\mathrm{op}} \subset H^{\mathrm{opcop}}$, since a left conditional expectation $E : H \to A$ is precisely a right conditional expectation $H^{\mathrm{op}} \to A^{\mathrm{op}}$, and since H is left A-faithfully flat if and only if H^{op} is right A^{op}-faithfully flat. $\square$

Notice that it is also possible to prove the previous result by adapting Section 3 to this left setting. We leave this to the reader. We have also the following variation of Theorem 1.1:

Theorem 4.2. *Let H be a Hopf algebra with bijective antipode and let $A \subset H$ be a right coideal subalgebra. If A is a direct summand in H as a left A-module, then H is faithfully flat as a left A-module.*

Proof. Apply Theorem 1.1 to the right coideal subalgebra $A^{\mathrm{op}} \subset H^{\mathrm{op}}$ (H^{op} being a Hopf algebra by bijectivity of the antipode). $\square$

4.2. **Projectivity.** One nice and useful feature of faithful flatness in the setting of coideal subalgebras is that it is often equivalent to projectivity [9]. Here this gives the following result.

Theorem 4.3. *Let H be a Hopf algebra with bijective antipode and let $A \subset H$ be a left or right coideal subalgebra.*

(1) *If A is a direct summand in H as a right A-module, then H is projective as a right A-module.*
(2) *If A is a direct summand in H as a left A-module, then H is projective as a left A-module.*

Proof. Assume first that $A \subset H$ is a right coideal subalgebra. (1) follows from Theorem 1.1 combined with the right version of [9, Theorem 2] (which, as explained after the statement of [9, Theorem 2], follows from the left statement applied to $A^{\mathrm{op}} \subset H^{\mathrm{op}}$). (2) follows from Theorem 4.2 and [9, Theorem 2]. The case when $A \subset H$ is a left coideal subalgebra follows from the right case applied to $A^{\mathrm{op}} \subset H^{\mathrm{opcop}}$. $\square$

4.3. **Example: free wreath products.** We finish the paper by an illustration. For $n \geq 1$, consider the algebra $A_s(n)$ presented by generators u_{ij}, $1 \leq i, j \leq n$, and relations

$$\sum_{j=1}^{n} u_{ij} = 1 = \sum_{j=1}^{n} u_{ji}, \quad u_{ij}u_{ik} = 0 = u_{ji}u_{ki}, \text{ for } k \neq j,$$

This is the coordinate algebra of Wang's quantum permutation group [21], which can be defined over any field, and has the Hopf algebra structure

$$\Delta(u_{ij}) = \sum_{k} u_{ik} \otimes u_{kj}, \ \varepsilon(u_{ij}) = \delta_{ij}, \ S(u_{ij}) = u_{ji}$$

Let A be a Hopf algebra, and consider A^{*n}, the free product algebra of n copies of A, which inherits a natural Hopf algebra structure such that the canonical morphisms $\nu_i : A \longrightarrow A^{*n}$, $1 \leq i \leq n$, are Hopf algebras morphisms. The free wreath product

$A *_w A_s(n)$ [3] is the quotient of the algebra $A^{*n} * A_s(n)$ by the two-sided ideal generated by the elements:

$$\nu_k(a)u_{ki} - u_{ki}\nu_k(a) \ , \quad 1 \leq i, k \leq n \ , \quad a \in A.$$

The free wreath product $A *_w A_s(n)$ admits a Hopf algebra structure given by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad \Delta(\nu_i(a)) = \sum_{k=1}^{n} \nu_i(a_{(1)})u_{ik} \otimes \nu_k(a_{(2)}),$$

$$\varepsilon(u_{ij}) = \delta_{ij}, \quad \varepsilon(\nu_i(a)) = \varepsilon(a), \ S(u_{ij}) = u_{ji}, \quad S(\nu_i(a)) = \sum_{k=1}^{n} \nu_k(S(a))u_{ki}.$$

It is immediate that there is an algebra map $E : A *_w A_s(n) \to A^{*n}$ given by $E(u_{ij}) = \delta_{ij}$ and $E(\nu_i(a)) = \nu_i(a)$, which thus gives a retraction to the natural map $A^{*n} \to A *_w A_s(n)$. Hence A^{*n} stands as left coideal subalgebra of $A *_w A_s(n)$, and E furnishes a left and right conditional expectation. Theorem 4.3 therefore ensures that if A has bijective antipode, then $A *_w A_s(n)$ is projective as a left and right A^{*n}-module. This was shown and used in the proof of [4, Theorem 8.4], using [7], under the additional assumption that $k = \mathbb{C}$ and that A is a compact Hopf algebra.

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