Militaru's D-Equation in Monoidal Categories

Julien Bichon - Ross Street

Abstract

A FRT type construction is done in a minimal categorical context: the ambient monoidal category is only assumed to have coequalizers. The early motivation for this construction was G. Militaru's work on the D-equation. We get generalizations of Militaru's constructions and results. The D-equation is also studied using the classical FRT construction: this leads to a notion of D-bialgebra. New solutions of the D-equation are constructed.

1 Introduction

Let M be a vector space over a field k and let $R \in \mathrm{End}_k(M \otimes M)$. One says that R is a solution of the D-equation if

$$(R \otimes 1_M) \circ (1_M \otimes R) = (1_M \otimes R) \circ (R \otimes 1_M).$$

We also say that R is a D-operator. This equation was named and studied by G. Militaru in [10]. It is explained in [11] that the D-equation is a part of the integrability condition for the Knizhnik-Zamolodchikov equation, see [6]. The aim of the present paper is to continue and extend Militaru's work [10].

We refer to the introduction of [10] for an overview of the use of bialgebra techniques in studying non-linear equations. To contrast Militaru's results (as well as ours) with the ones obtained previously in the case of the Yang-Baxter equation, let us first recall those results. Let M be a finite-dimensional k-vector space and let $R \in \mathrm{End}_k(M \otimes M)$.

(1) There exists a universal bialgebra $A(R)$ such that M is an $A(R)$-comodule and R is an $A(R)$-comodule morphism (this is the famous FRT bialgebra of Faddeev, Reshetikhin and Takhtajan [15]).

If R is a Yang-Baxter operator, there is more to be said about the bialgebra $A(R)$.

(2) The bialgebra $A(R)$ is coquasitriangular (= cobraided) and R arises from the braiding on the category of $A(R)$-comodules.

(3) R arises from a Yetter-Drinfeld module over $A(R)$.

Property **(2)** is due to Larson-Towber [7] while property **(3)** is due to Radford [14].

Let R be a D-operator on a finite-dimensional vector space. Militaru [10] constructs a universal bialgebra $D(R)$ satisfying an analogue of **(3)**: R arises from a Long dimodule over $D(R)$. However $D(R)$ does not satisfy anymore **(1)**: the quadratic relations of $A(R)$ are replaced by degree one relations. Also there is no analogue of **(2)**, i.e. there is no characterization by additional structure on the bialgebra $D(R)$. In section 4 of [10], Militaru considers coalgebras rather than bialgebras and tries to obtain an analogue of **(2)**, but there is a certain kind of "asymmetry" in the proposed definition, which only disappears for special D-operators (see (2) in Theorem 4.6 of [10]).

In this paper we develop two approaches to the D-equation. First we put Militaru's constructions and results in a minimal categorical framework: we do not assume the existence of a braiding on the ambient monoidal category $\mathcal{V}$ (in which coequalizers are assumed to exist). This means that we do not have a notion of bialgebra in $\mathcal{V}$, and so we work with coalgebras. This leads us to a new approach to the FRT construction. Let M be an object of $\mathcal{V}$, assumed to have a left dual, and let $R : M \otimes M \longrightarrow M \otimes M$ be a morphism. We construct a universal coalgebra $D(R)$ such that M is a $D(R)$-comodule and that $R : M_0 \otimes M \longrightarrow M_0 \otimes M$ is a $D(R)$-comodule morphism. Here M_0 denotes the trivial comodule associated to M and we exploit the fact that the category of comodules over a coalgebra in $\mathcal{V}$ is a $\mathcal{V}$-actegory [1, 12, 13, 9]. When R is a solution of the D-equation, we imitate Militaru's constructions to show that R arises from a Long comodule over $D(R)$. In this way our coalgebra satisfies analogues of both **(1)** and **(3)**. Miltaru's bialgebra $D(R)$ is a tensor algebra analogue of our coalgebra $D(R)$: see section 4 for a more precise comparison. The absence of a braiding forced us to modify slightly some of Militaru's formulas: these modifications are in fact simplifications.

We also use the classical FRT construction to study the D-equation. One motivation for this approach is the fact that certain D-operators, built from Frobenius algebras, arise naturally as morphisms in the representation categories of quantum automorphism groups [21, 2]. In this way we get bialgebras satisfying **(1)** and an analogue of **(2)**: we have a notion of D-bialgebra and of D-structure on a monoidal category.

Our work is organized as follows. In Section 2 we recall the classical FRT construction. Section 3 is devoted to the FRT construction in a minimal categorical framework. This general construction is used in the fourth Section to get analogues of Militaru's results. In Section 5 we use the classical FRT construction to study the D-equation. This leads to a notion of D-structure on a monoidal category and to a notion of D-bialgebra. In Section 6 some examples of D-operators arising from Frobenius algebras and finite groups are constructed.

2 The classical FRT construction

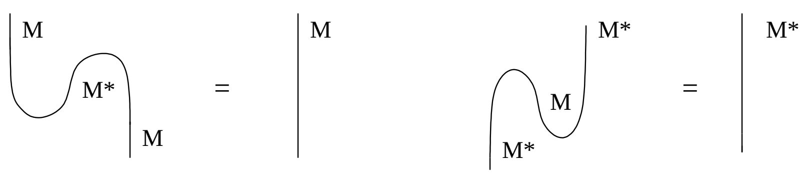
In this section we recall the classical FRT theorem. In order to introduce notation and techniques needed in section 5, we review the basic ideas of the proof. We assume the reader to be familiar with monoidal categories [8] with braided monoidal categories [5], and with the graphical calculus in monoidal categories [4]. We also assume familiarity with algebras, coalgebras, bialgebras (when the category is braided) and their modules and comodules in the framework of abstract monoidal categories. However let us recall that the category of (right) comodules over a bialgebra in a braided monoidal category, is, in a natural way, a monoidal category. Here is the classical FRT theorem :

Theorem 2.1 *Let $\mathcal{V}$ be a braided monoidal category which is cocomplete and such that*

the tensor product commutes with colimits. Let M be an object of $\mathcal{V}$ which has a left dual and let $R : M \otimes M \longrightarrow M \otimes M$ be a morphism of $\mathcal{V}$.
a) There exists a bialgebra $A(R)$ such that M is an $A(R)$-comodule and R is a morphism of $A(R)$-comodules.
b) If B is a bialgebra such that M is a B-comodule and R is a morphism of B-comodules, then there exists a unique morphism of bialgebras $\gamma : A(R) \longrightarrow B$ such that $(1_M \otimes \gamma) \circ \alpha = \beta$, where $\alpha : M \longrightarrow M \otimes A(R)$ and $\beta : M \longrightarrow M \otimes B$ denote the coactions of $A(R)$ and B respectively.

We review the ingredients of the proof. Let us first recall that a left dual for M is a triplet (M^*, e, d) where M^* is an object of $\mathcal{V}$, while $e : M^* \otimes M \longrightarrow I$ (I is the monoidal unit of $\mathcal{V}$) and $d : I \longrightarrow M \otimes M^*$ are morphisms of $\mathcal{V}$ such that:

$$(1_M \otimes e) \circ (d \otimes 1_M) = 1_M \quad \text{and} \quad (e \otimes 1_{M^*}) \circ (1_{M^*} \otimes d) = 1_{M^*}.$$



The full subcategory of $\mathcal{V}$ whose objects are objects with a left dual is denoted by $\mathcal{V}_0$.

The bialgebra $A(R)$ is constructed as the coendomorphism object of a functor. This process has a long history of generalization: [16], [20], [18], [3], [17],[13]. Let $\mathcal{C}$ be a (small) category, let $\mathcal{V}$ be a monoidal category and let $F : \mathcal{C} \longrightarrow \mathcal{V}_0$ be a functor. Reconstruction theory (Tannaka duality) associates an object coend(F) to such data.

• Reconstruction of a coalgebra. We assume that $\mathcal{V}$ is cocomplete. It turns out that the functor $\mathcal{V} \longrightarrow \text{Set}$, $V \longmapsto \text{Nat}(F, F \otimes V)$ is representable: there is an object A ($A = \text{coend}(F)$) of $\mathcal{V}$ and an element $\alpha \in \text{Nat}(F, F \otimes A)$ such that for every object V, the map

$$\begin{aligned} \theta_V : \text{Hom}_{\mathcal{V}}(A, V) &\longrightarrow \text{Nat}(F, F \otimes V) \\ f &\longmapsto (1_F \otimes f) \circ \alpha \end{aligned}$$

is bijective. Let $\Delta \in \text{Hom}_{\mathcal{V}}(A, A \otimes A)$ be such that $\theta_{A^{\otimes 2}}(\Delta) = (\alpha \otimes 1_A) \circ \alpha$ and let $\varepsilon \in \text{Hom}_{\mathcal{V}}(A, I)$ be such that $\theta_I(\varepsilon) = 1_F$: we get a coalgebra (A, Δ, ε). The functor F factorizes through the category of right comodules over A followed by the forgetful functor, and A is universal with this property.

• Reconstruction of a bialgebra. We assume that $\mathcal{V}$ satisfies the conditions of Theorem 2.1, that $\mathcal{C}$ is a monoidal category and that the functor F is monoidal. Then it turns out that $A = \text{coend}(F)$ is a bialgebra. More precisely, let $n \in \mathbb{N}^*$ and let $F^n : \mathcal{C}^n \longrightarrow \mathcal{V}$ be the functor defined by $F^n(X_1,\ldots,X_n) = F(X_1)\otimes\ldots\otimes F(X_n)$: For all objects V of $\mathcal{V}$ the map

$$\theta^n_V : \mathrm{Hom}_{\mathcal{V}}(A^{\otimes n}, V) \longrightarrow \mathrm{Nat}(F^n, F^n\otimes V)$$
$$f \longmapsto \theta^n_V(f)_{X_1,\ldots,X_n} = (1_{F(X_1)}\otimes\ldots\otimes 1_{F(X_n)}\otimes f)\circ\tau\circ\alpha_{X_1}\otimes\ldots\otimes\alpha_{X_n}$$

where τ is the obvious morphism arising from the braid group $\mathbb{B}_{2n}$, is bijective. For example $\theta^2_V(f)_{X,Y} = (1_{F(X)}\otimes 1_{F(Y)}\otimes f)(1_{F(X)}\otimes c_{A,F(Y)}\otimes 1_A)\circ(\alpha_X\otimes\alpha_Y)$ where c is the braiding of $\mathcal{V}$. The product of A is now defined by $\theta^2_A(m)_{X,Y} = (\widetilde{F}^{-1}_{X,Y}\otimes 1_A)\circ\alpha_{X\otimes Y}\circ\widetilde{F}_{X,Y}$ where the isomorphisms $\widetilde{F}_{X,Y} : F(X)\otimes F(Y)\cong F(X\otimes Y)$ are part of the monoidal functor F. The functor F factorizes through a monoidal functor with values in the category of A-comodules, and the bialgebra A is universal with this property.

We come back to the construction of $A(R)$. We consider the following category $\mathbb{M}$. The objects are positive integers $0,1,2\ldots$; the homsets are given by $\mathbb{M}(n,m)=\emptyset$ if $n\neq m$ and $\mathbb{M}(n,n)=\mathbb{M}_n$ is the free monoid generated by $n-1$ indeterminates $s_1,\ldots,s_{n-1}$ and submitted to the relations $s_is_j = s_js_i$ if $|i-j|>2$. The category $\mathbb{M}$ is equipped with a strict monoidal structure defined on the objects by $n\otimes m = n+m$ and by $s_i\otimes s_j = s_is_{n+j}$ for $s_i\in\mathbb{M}_n$ and $s_j\in\mathbb{M}_m$. Following the proof of Proposition 2.2 in [5], we have:

Proposition 2.2 *Let M be an object in a (strict) monoidal category $\mathcal{V}$ and let $R : M\otimes M\longrightarrow M\otimes M$ be a morphism of $\mathcal{V}$. Then there exists a unique strict monoidal functor $F_R : \mathbb{M}\longrightarrow\mathcal{V}$ such that $F_R(1) = M$ and $F_R(s_1) = R$.* □

Coming back to Theorem 2.1, we let $A(R) = \mathrm{coend}(F_R)$: M is an $A(R)$-comodule and $R : M\otimes M\longrightarrow M\otimes M$ is an $A(R)$-comodule morphism. The universal property described at the end of the theorem follows from the one of $\mathrm{coend}(F_R)$.

3 A minimal FRT construction

Let $\mathcal{V}$ be a monoidal category, let M be an object of $\mathcal{V}$ and let $R : M\otimes M\longrightarrow M\otimes M$ be a morphism. We would like to have an analogue of Theorem 2.1. However if we do not assume the presence of a braiding on $\mathcal{V}$, we do not have a notion of bialgebra. Even if M is a comodule over some coalgebra, there is no canonical comodule structure on $M\otimes M$. These difficulties lead us to consider the notion of monoidal category "acting" on a category. We shall use the term $\mathcal{V}$-actegory of [9] for this concept, which goes back at least to [1]. The terminology is $\mathcal{V}$-category in [12, 13].

Let $C = (C,\Delta,\varepsilon)$ be a coalgebra in $\mathcal{V}$. The category $\mathcal{C} = \mathrm{Comod}(C)$ is not a monoidal category, but a $\mathcal{V}$-actegory [1, 12, 13, 9]: we have a bifunctor $\mathcal{V}\times\mathcal{C}\longrightarrow\mathcal{C}$ with associativities constraints. Indeed if X is an object of $\mathcal{V}$ and M is a C-comodule with coaction $\delta : M\longrightarrow M\otimes C$, then $X\otimes M$ is a C-comodule with coaction $1_X\otimes\delta$.

To any C-comodule M, we associate, via the forgetful functor $\mathcal{C}\longrightarrow\mathcal{V}$ the underlying object, which we denote by M_0. Then we can consider the C-comodule $M_0\otimes M$ and if $R : M\otimes M\longrightarrow M\otimes M$ is a morphism we can ask R to be a C-comodule morphism

$M_0 \otimes M \longrightarrow M_0 \otimes M$. Let us record the following result, which is immediate from the definitions:

Proposition 3.1 *Let C be a coalgebra in the monoidal category $\mathcal{V}$, let M be a C-comodule with coaction $\delta : M \longrightarrow M \otimes C$ and let $R : M \otimes M \longrightarrow M \otimes M$ be a morphism of $\mathcal{V}$. Then $R : M_0 \otimes M \longrightarrow M_0 \otimes M$ is a C-comodule morphism if and only if $(1_M \otimes \delta) \circ R = (R \otimes 1_M) \circ (1_M \otimes \delta)$.* $\square$

After these considerations, it is easy to imagine the statement of our FRT theorem. First let us state an easy well-known lemma, whose proof is left to the reader.

Lemma 3.2 *Let $C = (C, \Delta_0, \varepsilon_0)$ be a coalgebra in a monoidal category $\mathcal{V}$, and let D be an object of $\mathcal{V}$ endowed with morphisms $\Delta : D \longrightarrow D \otimes D$ and $\varepsilon : D \longrightarrow I$. Let $\pi : C \longrightarrow D$ be an epimorphism such that $\Delta \circ \pi = (\pi \otimes \pi) \circ \Delta_0$. Then $D = (D, \Delta, \varepsilon)$ is a coalgebra. If M is a C-comodule with coaction $\delta : M \longrightarrow M \otimes C$, then M is a D-comodule with coaction $\delta = (1_M \otimes \pi) \circ \delta_0$.* $\square$

Here is our FRT type theorem:

Theorem 3.3 *Let $\mathcal{V}$ be a monoidal category in which coequalizers exist, let M be an object of $\mathcal{V}$ having a left dual and let $R : M \otimes M \longrightarrow M \otimes M$ be a morphism of $\mathcal{V}$.*
a) There exists a coalgebra $D(R)$ and a morphism $\delta : M \longrightarrow M \otimes D(R)$ making M a $D(R)$-comodule such that $R : M_0 \otimes M \longrightarrow M_0 \otimes M$ is a $D(R)$-comodule morphism.
b) The coalgebra $D(R)$ is universal in the following sense. Let V be an object of $\mathcal{V}$ and let $\delta' : M \longrightarrow M \otimes V$ be a morphism such that $(1_M \otimes \delta') \circ R = (R \otimes 1_V) \circ (1_{M'} \otimes \delta')$: then there exists a unique morphism $\gamma : D(R) \longrightarrow V$ such that $(1_M \otimes \gamma) \circ \delta = \delta'$. If furthermore V is a coalgebra and δ' makes M a V-comodule, then γ is morphism of coalgebras.

Proof. The proof is divided into several steps. The dual of M is denoted by (M^*, e, d) (see Section 2).
Step 1. We construct the object $D(R)$. Let us consider the coalgebra $C = M^* \otimes M$, with comultiplication $\Delta_0 = 1_{M^*} \otimes d \otimes 1_M$ and counit e. Let us now define several important morphisms:

$\mu_0 := (e \otimes 1_M) \circ (1_{M^*} \otimes R) : C \otimes M \longrightarrow M$,

$f := 1_{M^*} \otimes \mu_0 : M^* \otimes C \otimes M \longrightarrow C$,

$g := (e \otimes 1_C) \circ (f \otimes 1_C) \circ (1_{M^*} \otimes 1_C \otimes d \otimes 1_M) : M^* \otimes C \otimes M \longrightarrow C$.

We now define $D(R)$ to be the coequalizer of f and g :

$$M^* \otimes C \otimes M \underset{g}{\overset{f}{\rightrightarrows}} C \xrightarrow{\pi} D(R).$$

For simplicity we put $D = D(R)$.
Step 2. We define the coalgebra structure on D. Figure 2 shows that

$$(\pi \otimes \pi) \circ \Delta_0 \circ f = (\pi \otimes \pi) \circ \Delta_0 \circ g.$$

Hence there exists a unique morphism $\Delta : D \longrightarrow D \otimes D$ such that $\Delta \circ \pi = (\pi \otimes \pi) \circ \Delta_0$.

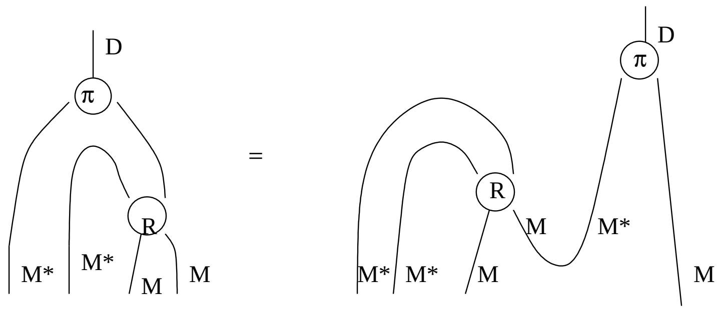


Figure 1: $\pi \circ f = \pi \circ g$

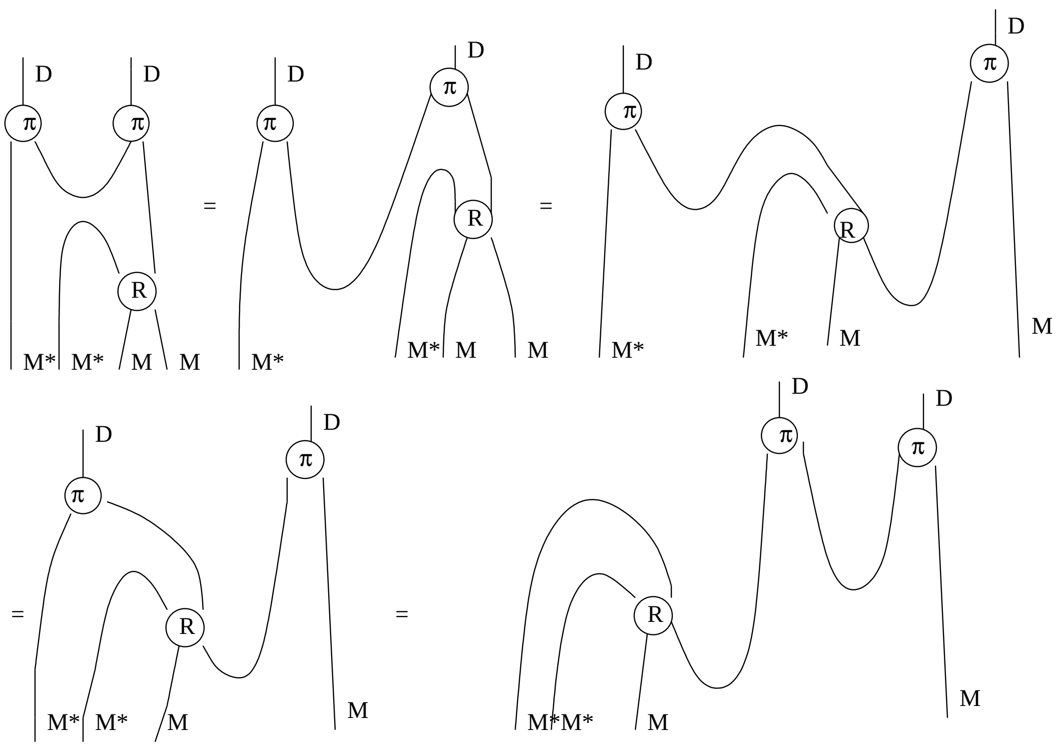


Figure 2: $(\pi \otimes \pi) \circ \Delta_0 \circ f = (\pi \otimes \pi) \circ \Delta_0 \circ g$

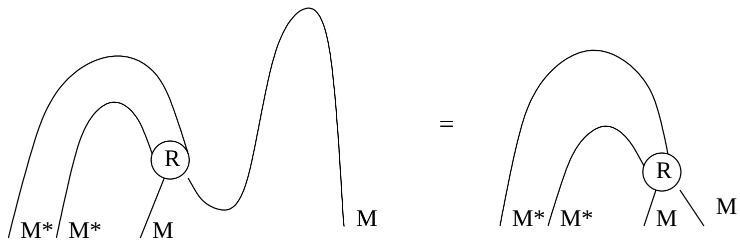


Figure 3: $e \circ g = e \circ f$

Let us now define the counit of D. One has $e \circ g = e \circ f$ by figure 3, hence there exists a unique morphism $\varepsilon : D \longrightarrow I$ such that $\varepsilon \circ \pi = e$. Lemma 3.2 ensures that (D, Δ, ε) is a coalgebra.

Step 3. We define a coaction of D on M. Let $\delta := (1_M \otimes \pi) \circ (d \otimes 1_M) : M \longrightarrow M \otimes D$. Since $\delta_0 = d \otimes 1_M$ is a coaction of C on M, it follows from Lemma 3.2 that δ is a coaction of D on M, and hence M is a D-comodule.

Step 4. We wish to prove that $R : M_0 \otimes M \longrightarrow M_0 \otimes M$ is a D-comodule morphism: that is, $(1_M \otimes \delta) \circ R = (R \otimes 1_D) \circ (1_M \otimes \delta)$.

Let us consider an arbitrary object V in $\mathcal{V}$, a morphism $\delta' : M \longrightarrow M \otimes V$ and the adjunction isomorphism:

$$\begin{aligned} \varphi : \mathrm{Hom}_{\mathcal{V}}(M \otimes M, M \otimes M \otimes V) &\longrightarrow \mathrm{Hom}_{\mathcal{V}}(M^* \otimes C \otimes M, V) \\ h &\longmapsto (e \otimes 1_V) \circ (1_{M^*} \otimes e \otimes 1_M \otimes 1_V) \circ (1_{M^*} \otimes 1_{M^*} \otimes h). \end{aligned}$$

Then by Figure 4, we have:

$$\varphi((1_M \otimes \delta') \circ R) = (e \otimes 1_V) \circ (1_{M^*} \otimes \delta') \circ f.$$

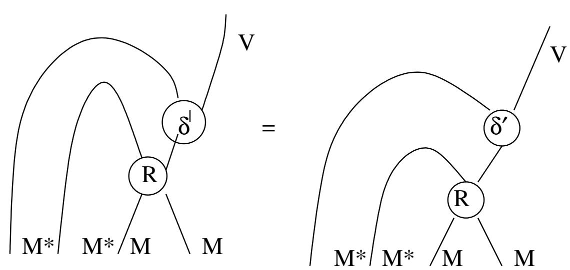


Figure 4: $\varphi((1_M \otimes \delta') \circ R) = (e \otimes 1_V) \circ (1_{M^*} \otimes \delta') \circ f$

On the other hand, Figure 5 shows that:

$$\varphi((R \otimes 1_V) \circ (1_M \otimes \delta')) = (e \otimes 1_V) \circ (1_{M^*} \otimes \delta') \circ g.$$

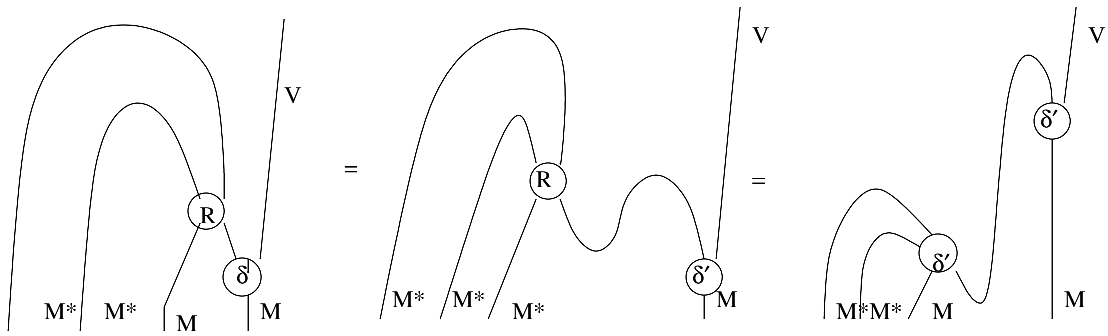


Figure 5: $\varphi((R \otimes 1_V) \circ (1_M \otimes \delta')) = (e \otimes 1_V) \circ (1_{M^*} \otimes \delta') \circ g$

Now if $V = D$ and $\delta' = \delta = (1_M \otimes \pi) \circ (d \otimes 1_M)$, then

$$(e \otimes 1_V) \circ (1_{M^*} \otimes \delta) = (e \otimes 1_V) \circ (1_{M^*} \otimes 1_M \otimes \pi) \circ (1_{M^*} \otimes d \otimes 1_M) = \pi.$$

Thus $\varphi((1_M \otimes \delta) \circ R) = \pi \circ f = \pi \circ g = \varphi((R \otimes 1_D) \circ (1_M \otimes \delta'))$, and we have $(1_M \otimes \delta) \circ R = (R \otimes 1_D) \circ (1_M \otimes \delta)$: part a) is proved.

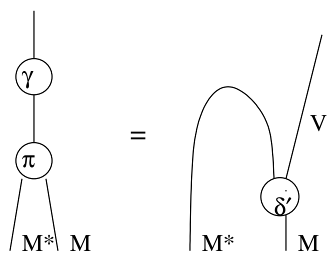


Figure 6: $\gamma \circ \pi = (e \otimes 1_V) \circ (1_{M^*} \otimes \delta')$

To prove part b), let us come back to an arbitrary object V and a morphism $\delta' : M \longrightarrow M \otimes V$ such that $(1_M \otimes \delta') \circ R = (R \otimes 1_V) \circ (1_M \otimes \delta')$. Then by the previous computations $(e \otimes 1_V) \circ (1_{M^*} \otimes \delta') \circ f = (e \otimes 1_V) \circ (1_{M^*} \otimes \delta') \circ g$. This means that there exists a unique morphism $\gamma : D \longrightarrow V$ such that $\gamma \circ \pi = (e \otimes 1_V) \circ (1_{M^*} \otimes \delta')$ (Figure 6) which implies (Figure 7)

$$(1_M \otimes \gamma) \circ (1_M \otimes \pi) = \delta',$$

as claimed. The morphism γ is clearly unique with this property.

If now $V = (V, \Delta_V, \varepsilon_V)$ is a coalgebra and δ' makes M into a V-comodule, by Figure 8 we have:

$$(\gamma \otimes \gamma) \circ \Delta \circ \pi = \Delta_V \circ \gamma \circ \pi.$$

This implies that $(\gamma \otimes \gamma) \circ \Delta \circ = \Delta_V \circ \gamma$. We also have $\varepsilon_V \circ \gamma \circ \pi = e = \varepsilon_V \circ \pi$ by Figure 9. This implies that $\varepsilon \circ \gamma = \varepsilon$: so γ is a coalgebra morphism. □

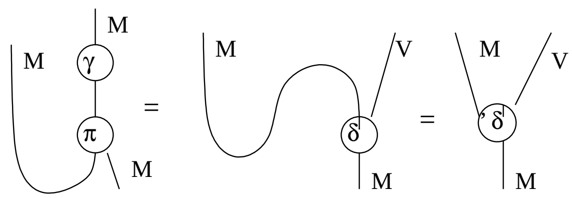


Figure 7: $(1_M \otimes \gamma) \circ \delta = \delta'$

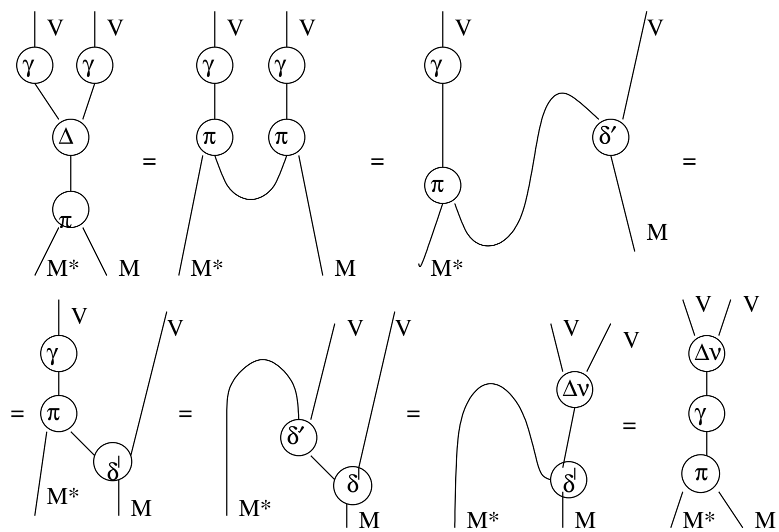


Figure 8: $(\gamma \otimes \gamma) \circ \Delta \circ \pi = \Delta_V \circ \gamma \circ \pi$

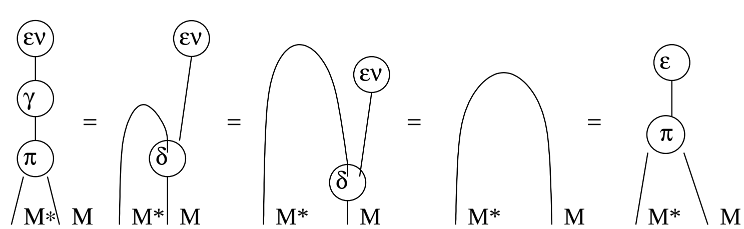


Figure 9: $\varepsilon_V \circ \gamma \circ \pi = \varepsilon \circ \pi$

Example 3.4 Let k be a commutative field and let M be a finite-dimensional vector space over k with basis $(e_i)_{1\le i\le n}$ and dual basis $(e_i^*)_{1\le i\le n}$. Let $R : M\otimes M \longrightarrow M\otimes M$ be a k-linear operator with $R(e_i\otimes e_j) = \sum_{k,l} R_{kl}^{ij}\ e_k\otimes e_l$. Then the coalgebra $D(R)$ is the quotient of the coalgebra $M^*\otimes M$ by the linear subspace generated by the elements:

$$\sum_{s=1}^{n} R_{ks}^{ij}\ e_l^*\otimes e_s - \sum_{s=1}^{n} R_{kl}^{is}\ e_s^*\otimes e_j, \quad 1\le i,j,k,l\le n.$$

We close this section by noting the fact that a dual of Theorem 3.3 can be stated with the use of algebras instead of coalgebras, if the category has equalizers instead of coequalizers. Indeed if A is an algebra in $\mathcal{V}$, the category of left A-modules in $\mathcal{V}$ is a right $\mathcal{V}$-actegory, and we can ask a morphism $R : M\otimes M \longrightarrow M\otimes M$ to be an A-module morphism $R : M\otimes M_0 \longrightarrow M\otimes M_0$. If M has a left dual and the category has equalizers, a dual theorem to Theorem 3.3 can be proved in the same way. In Example 3.4, we get the dual algebra of $D(R)$, that is the set of elements f of $\mathrm{End}_k(M)$ such that $(f\otimes 1_M)\circ R = R\circ(f\otimes 1_M)$.

4 The D-equation and Long comodules

We now come to the D-equation, our initial motivation for considering the FRT type construction of the previous section. The results obtained here put Militaru's work [10] in a categorical framework. We work over a fixed monoidal category $\mathcal{V}$.

Definition 4.1 *Let M be an object of $\mathcal{V}$ and let $R : M\otimes M \longrightarrow M\otimes M$ be a morphism. Then we say that R is a solution of the D-equation if*

$$(R\otimes 1_M)\circ(1_M\otimes R) = (1_M\otimes R)\circ(R\otimes 1_M).$$

We also say that R is a D-morphism

The prefix "D" was coined by Militaru, referring to Long dimodules. Here we will deal with Long comodules, but we keep Militaru's terminology. The following definition is inspired by [10], Definition 3.1.

Definition 4.2 *a) A Long system $\mathcal{L} = (M, V, \delta, \mu)$ consists of objects M and V of $\mathcal{V}$, and of morphisms $\delta : M \longrightarrow M\otimes V$ and $\mu : V\otimes M \longrightarrow M$ such that*

$$\delta\circ\mu = (\mu\otimes 1_V)\circ(1_V\otimes\delta).$$

b) Let C be a coalgebra. A Long C-comodule (M, δ, μ) consists of a C-comodule (M, δ) and of a morphism $\mu : C\otimes M \longrightarrow M$ such that (M, C, δ, μ) is a Long system.

The following result is the analogue of [10], Proposition 3.4. It also establishes a connection between the considerations of paragraph 3 and the D-equation.

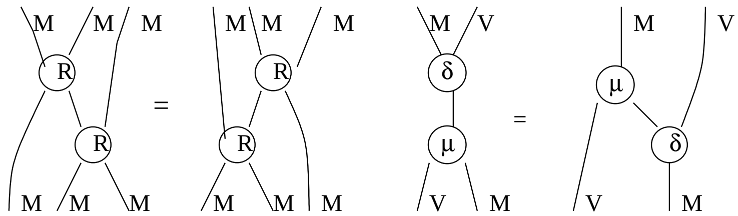


Figure 10: The D-equation; A Long system

Proposition 4.3 *Let $\mathcal{L} = (M, V, \delta, \mu)$ be a Long system. Then $R = R_{\mathcal{L}} := (1_M \otimes \mu) \circ (\delta \otimes 1_M)$ satisfies*

$$(1_M \otimes \delta) \circ R = (R \otimes 1_V) \circ (1_M \otimes \delta)$$

and is a solution of the D-equation.

Proof. We have:

$$\begin{aligned}(1_M \otimes \delta) \circ R &= (1_M \otimes \delta) \circ (1_M \otimes \mu) \circ (\delta \otimes 1_M) \\ &= (1_M \otimes \mu \otimes 1_V) \circ (1_M \otimes 1_V \otimes \delta) \circ (\delta \otimes 1_M) \\ &= ((1_M \otimes \mu) \circ (\delta \otimes 1_M)) \otimes 1_V \circ (1_M \otimes \delta) \\ &= (R \otimes 1_V) \circ (1_M \otimes \delta).\end{aligned}$$

Thus we get:

$$\begin{aligned}(1_M \otimes R) \circ (R \otimes 1_M) &= (1_M \otimes 1_M \otimes \mu) \circ (1_M \otimes \delta \otimes 1_M) \circ R \otimes 1_M \\ &= (1_M \otimes 1_M \otimes \mu) \circ (R \otimes 1_V \otimes 1_M) \circ (1_M \otimes \delta \otimes 1_M) \\ &= (R \otimes 1_M) \circ (1_M \otimes R). \quad \square\end{aligned}$$

We shall provide a converse to Proposition 4.3. We first prove a lemma. The notation was introduced in the proof of Theorem 3.3.

Lemma 4.4 *Let M be an object of $\mathcal{V}$ having a left dual and let $R : M \otimes M \longrightarrow M \otimes M$ be a morphism. The coalgebra $M^* \otimes M$ is denoted by C. Let*
$\mu_0 := (e \otimes 1_M) \circ (1_{M^*} \otimes R) : C \otimes M \longrightarrow M,$
$f := 1_{M^*} \otimes \mu_0 : M^* \otimes C \otimes M \longrightarrow C,$
$g := (e \otimes 1_C) \circ (f \otimes 1_C) \circ (1_{M^*} \otimes 1_C \otimes d \otimes 1_M) : M^* \otimes C \otimes M \longrightarrow C.$
Then R is a solution of the D-equation if and only if

$$\mu_0 \circ (f \otimes 1_M) = \mu_0 \circ (g \otimes 1_M).$$

Proof. Let us consider the adjunction isomorphism:

$$\begin{aligned}\psi : \mathrm{Hom}_{\mathcal{V}}(M^{\otimes 3}, M^{\otimes 3}) &\longrightarrow \mathrm{Hom}_{\mathcal{V}}((M^*)^{\otimes 2} \otimes M^{\otimes 3}, M) \\ h &\longmapsto (e \otimes 1_M) \circ (1_{M^*} \otimes e \otimes 1_M \otimes 1_M) \circ (1_{M^*} \otimes 1_{M^*} \otimes h).\end{aligned}$$

Then Figure 11 shows that

$$\psi((1_M \otimes R) \circ (R \otimes 1_M)) = \mu_0 \circ (f \otimes 1_M) \quad \text{and}$$
$$\psi((R \otimes 1_M) \circ (1_M \otimes R)) = \mu_0 \circ (g \otimes 1_M).$$

The assertion of the lemma is now clear. □

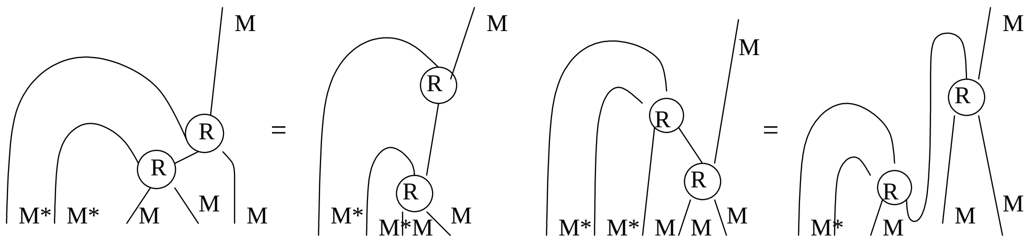


Figure 11: Proof of Lemma 4.4

Here is our analogue of Militaru's Theorem 3.6 in [10]:

Theorem 4.5 *Let $\mathcal{V}$ be a monoidal category in which coequalizers exist, let M be an object of $\mathcal{V}$ having a left dual and let $R : M \otimes M \longrightarrow M \otimes M$ be a D-morphism. Consider the coalgebra $D(R)$ of Theorem 3.3 and the coaction $\delta : M \longrightarrow M \otimes D(R)$.*
a) There exists a morphism $\mu : D(R) \otimes M \longrightarrow M$ such that (M, δ, μ) is a Long $D(R)$-comodule and $R = R_{\mathcal{L}}$, where $\mathcal{L} = (M, D(R), \delta, \mu)$ is the corresponding Long system.
b) The coalgebra $D(R)$ has the following universal property. Let $\mathcal{L}' = (M, V, \delta', \mu')$ be a Long system such that $R = R_{\mathcal{L}'}$: then there exists a unique morphism $\gamma : D(R) \longrightarrow V$ such that $(1_M \otimes \gamma) \circ \delta = \delta'$ and $\mu' \circ (\gamma \otimes 1_M) = \mu$. If furthermore V is a coalgebra such that (M, δ', μ') is a Long V-comodule, then γ is a morphism of coalgebras.

Proof. a) We use the notation of the proof of Theorem 3.3 and Lemma 4.4. Let us construct the morphism $\mu : D \otimes M \longrightarrow M$. The functor $- \otimes M$ has a right adjoint, and so preserves coequalizers. Thus $\pi \otimes 1_M : C \otimes M \longrightarrow D \otimes M$ is the coequalizer of

$$M^* \otimes C \otimes M \otimes M \underset{g \otimes 1_M}{\overset{f \otimes 1_M}{\rightrightarrows}} C \otimes M.$$

By Lemma 4.4 one has $\mu_0 \circ (f \otimes 1_M) = \mu_0 \circ (g \otimes 1_M)$. Thus there exists a unique morphism $\mu : D \otimes M \longrightarrow M$ such that $\mu \circ (\pi \otimes 1_M) = \mu_0$. Let us check that (M, D, δ, μ) is a Long system. By Figure 12 one has

$$\delta \circ \mu \circ (\pi \otimes 1_M) = (\mu \otimes 1_D) \circ (1_D \otimes \delta) \circ (\pi \otimes 1_M).$$

Hence $\delta \circ \mu = (\mu \otimes 1_D) \circ (1_D \otimes \delta)$ since $\pi \otimes 1_M$ is an epimorphism: $\mathcal{L} = (M, D, \delta, \mu)$ is a Long system. Then

$$\begin{aligned} R_{\mathcal{L}} &= (1_M \otimes \mu) \circ (\delta \otimes 1_M) \\ &= (1_M \otimes \mu) \circ (1_M \otimes \pi \otimes 1_M) \circ (d \otimes 1_M \otimes 1_M) \\ &= (1_M \otimes \mu_0) \circ (d \otimes 1_M \otimes 1_M) \\ &= (1_M \otimes e \otimes 1_M) \circ (1_M \otimes 1_{M^*} \otimes R) \circ (d \otimes 1_M \otimes 1_M) = R. \end{aligned}$$

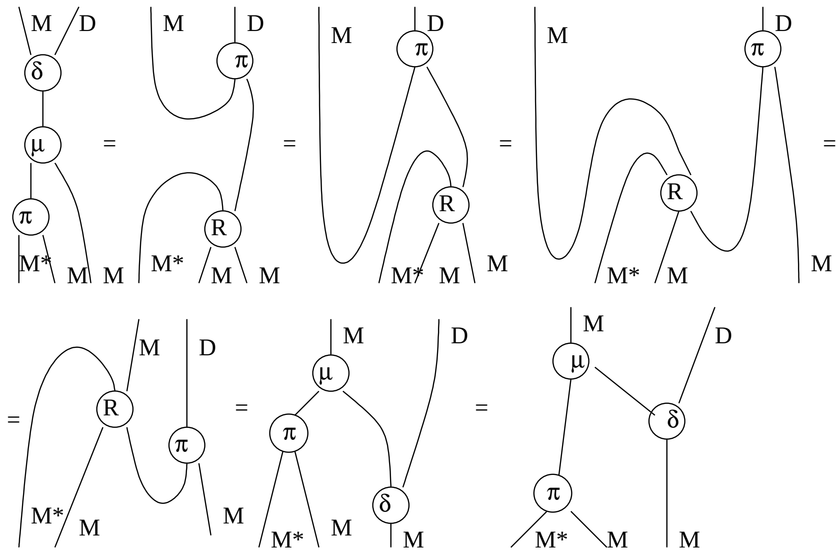


Figure 12: $\delta \circ \mu \circ (\pi \otimes 1_M) = (\mu \otimes 1_D) \circ (1_D \otimes \delta) \circ (\pi \otimes 1_M)$

b) Assume that $R = R_{\mathcal{L}'}$ where $\mathcal{L}' = (M, V, \delta', \mu')$ is a Long system. Then by Proposition 4.3, $(1_M \otimes \delta') \circ R = (R \otimes 1_V) \circ (1_M \otimes \delta')$. Thus by Theorem 3.3, there exists a unique morphism $\gamma : D \longrightarrow M$ such that $(1_M \otimes \gamma) \circ \delta = \delta'$, and we also have $\gamma \circ \pi = (e \otimes 1_V) \circ (1_{M^*} \otimes \delta')$. Then

$$
\begin{aligned}
\mu' \circ (\gamma \otimes 1_M) \circ (\pi \otimes 1_M) &= \mu' \circ (e \otimes 1_V \otimes 1_M) \circ (1_{M^*} \otimes \delta' \otimes 1_M) \\
&= (e \otimes 1_M) \circ (1_{M^*} \otimes 1_M \otimes \mu') \circ (1_{M^*} \otimes \delta' \otimes 1_M) \\
&= (e \otimes 1_M) \circ (1_{M^*} \otimes R) = \mu_0 = \mu \circ (\pi \otimes 1_M)
\end{aligned}
$$

and hence $\mu' \circ (\gamma \otimes 1_M) = \mu$ since $\pi \otimes 1_M$ is an epimorphism. The last assertion follows immediately from Theorem 3.3. □

It is now time for a definitive comparison between our constructions and those of Militaru [10]. The absence of a braiding led us to modifications in some of Militaru's formulas. The following proposition establishes the connection between both constructions, and also gives a method of producing new D-morphisms from old ones.

Proposition 4.6 *Let $\mathcal{V}$ be a braided monoidal category with braiding c and let M be an object of $\mathcal{V}$. Let $R : M \otimes M \longrightarrow M \otimes M$ be a D-morphism. Then $\tilde{R} := c^{-1}_{M,M} \circ R \circ c_{M,M}$ is also a D-morphism.*

Proof. The proof is done by Figure 13. □

We can now make our comparison. Militaru [10] associates the operator $\tilde{R_{\mathcal{L}}}$ to a Long system $\mathcal{L}$. The coalgebra $C(R)$ of paragraph 4 in [10] is our coalgebra $D(\tilde{R})$, while the bialgebra $D(R)$ of paragraph 3 in [10] is a tensor algebra analogue of our $D(\tilde{R})$.

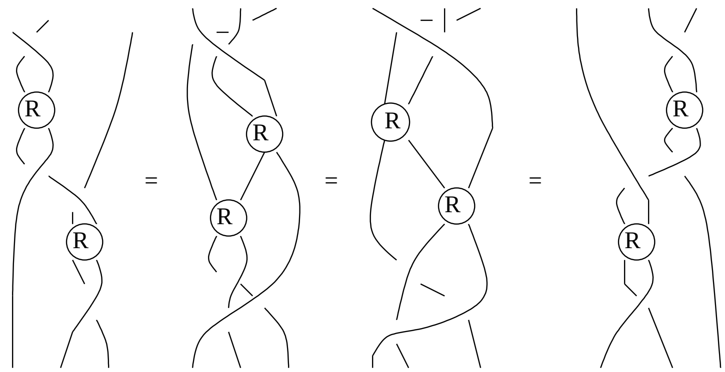


Figure 13: $(\tilde{R} \otimes 1_M) \circ (1_M \otimes \tilde{R}) = (1_M \otimes \tilde{R}) \circ (\tilde{R} \otimes 1_M)$

5 A notion of D-bialgebra

We now wish to study the D-equation using the classical FRT constuction. The first reason is that we would like to have a characterization of bialgebras furnishing solutions of the D-equation, just as coquasitriangular bialgebras are the bialgebras furnishing solutions of the quantum Yang-Baxter equation. This leads us to a notion of D-structure on a monoidal category, and to a notion of D-bialgebra. The second motivation for using the classical FRT construction is that it turns out that certain D-morphisms, constructed from Frobenius algebras, arise as morphisms in the representation categories of recently constructed quantum automorphism groups in [21] and [2]. With this observation, the classical FRT bialgebras are much more convenient. This topic will be treated in the next section.

Definition 5.1 *Let $\mathcal{V}$ be a (strict) monoidal category. A D-structure on $\mathcal{V}$ consists of a family of morphisms $R_M : M \longrightarrow M$, natural in M such that*

$$(R_{M\otimes N} \otimes 1_P) \circ (1_M \otimes R_{N\otimes P}) = (1_M \otimes R_{N\otimes P}) \circ (R_{M\otimes N} \otimes 1_P)$$

for all objects M, N, P, and $R_I = 1_I$ (where I denotes the monoidal unit).

As a first example, we take a look at the free monoidal category generated by a D-morphism. We consider the following category $\mathbb{D}$. The objects are positive integers $0, 1, 2\dots$; the homsets are given by $\mathbb{D}(n,m) = \emptyset$ if $n \neq m$ and $\mathbb{D}(n,n) = \mathbb{D}_n$ is the free abelian monoid generated by $n-1$ indeterminates $s_1, \dots, s_{n-1}$. The category $\mathbb{D}$ is equipped with a strict monoidal structure defined on the objects by $n \otimes m = n+m$ and by $s_i \otimes s_j = s_i s_{n+j}$ for $s_i \in \mathbb{D}_n$ and $s_j \in \mathbb{D}_m$. Following once again the proof of Proposition 2.2 in [5], we have:

Proposition 5.2 *Let M be an object in a (strict) monoidal category $\mathcal{V}$ and let $R : M \otimes M \longrightarrow M \otimes M$ be a D-morphism. Then there exists a unique strict monoidal functor $F_R : \mathbb{D} \longrightarrow \mathcal{V}$ such that $F_R(1) = M$ and $F_R(s_1) = R$.* $\square$

One defines a D-structure on the category $\mathbb{D}$ by choosing, for any n, an element R_n of $\mathbb{D}_n$. It is clear that the most interesting case occurs when one takes $R_2 = s_1$.

We now work in a (strict) braided monoidal category $\mathcal{V} = (\mathcal{V}, c)$. Let us recall that a bialgebra $A = (A, m, u, \Delta, \varepsilon)$ is both an algebra (A, m, u) and a coalgebra (A, Δ, ε) in $\mathcal{V}$, such that $\Delta : A \longrightarrow A \otimes A$ are algebras morphisms, that is:

$$\Delta \circ m = (m \otimes m) \circ (1_A \otimes c_{A,A} \otimes 1_A) \circ \Delta \quad \text{and} \quad \varepsilon \circ m = \varepsilon \otimes \varepsilon.$$

The opposite comultiplication is $\Delta^0 = c_{A,A}^{-1} \circ \Delta$.

Definition 5.3 *Let A be a bialgebra in the braided monoidal category $\mathcal{V}$. A D-morphism on A is a morphism $\phi : A \longrightarrow I$ such that*

$$(\phi \otimes \phi) \circ (m \otimes m) \circ (1_A \otimes \Delta \otimes 1_A) = (\phi \otimes \phi) \circ (m \otimes m) \circ (1_A \otimes \Delta^0 \otimes 1_A),$$

$$(\phi \otimes 1_A) \circ \Delta = (1_A \otimes \phi) \circ \Delta \quad \text{and} \quad \phi \circ u = 1_I.$$

A D-bialgebra (A, ϕ) consists of a bialgebra A and of a D-morphism ϕ on A.

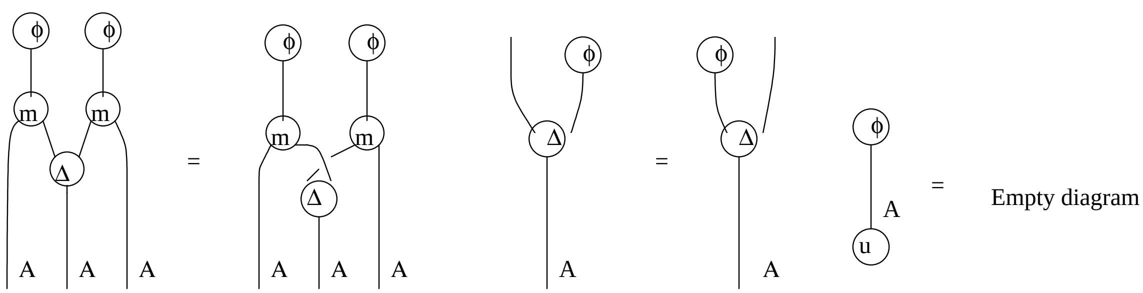


Figure 14: Axioms for a D-bialgebra

Let k be a commutative field and let A be a bialgebra over k. Using Sweedler's notations [19, 6], we see that a linear map $\phi : A \longrightarrow k$ is a D-morphism if and only if, for all $x, y, z \in k$, we have

$$\sum \phi(xy_{(1)})\phi(y_{(2)}z) = \sum \phi(xy_{(2)})\phi(y_{(1)}z),$$

$$\sum \phi(x_{(1)})x_{(2)} = \phi(x_{(2)})x_{(1)} \quad \text{and} \quad \phi(1) = 1.$$

The second condition means that ϕ is central in the convolution algebra $\mathrm{Hom}_k(A, k)$.

Proposition 5.4 *Let (A, ϕ) be D-bialgebra in the braided monoidal category $\mathcal{V}$. Then the category* Comod*(A) of comodules over A inherits a D-structure defined, for an A-comodule M with coaction α_M, by $R_M = (1_M \otimes \phi) \circ \alpha_M$.*

Proof. It is easily seen that the family of morphism R_M is natural, that $R_I = 1_I$ and that each R_M is a morphism of A-comodules. It remains to check that for all comodules M,N, P, one has:

$$(R_{M\otimes N} \otimes 1_P) \circ (1_M \otimes R_{N\otimes P}) = (1_M \otimes R_{N\otimes P}) \circ (R_{M\otimes N} \otimes 1_P).$$

This is done by Figure 15, where ψ stands for $\phi \circ m$.□

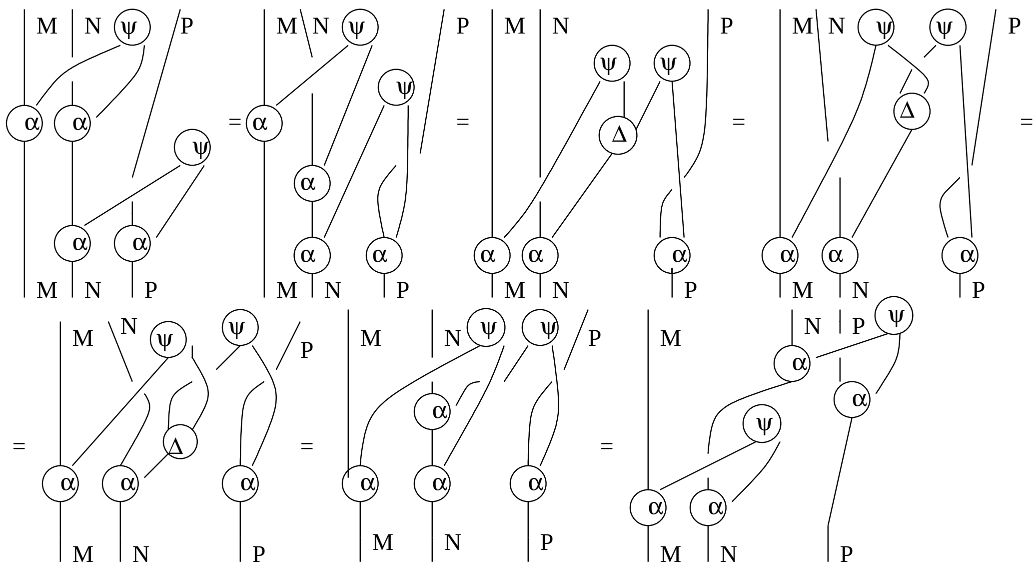


Figure 15: $(R_{M\otimes N} \otimes 1_P) \circ (1_M \otimes R_{N\otimes P}) = (1_M \otimes R_{N\otimes P}) \circ (R_{M\otimes N} \otimes 1_P)$

It is classical in reconstruction theory to have a converse for a result such as Proposition 5.4. Indeed we have:

Proposition 5.5 *Let $\mathcal{C}$ be a monoidal category equipped with a D-structure, let $\mathcal{V}$ be a cocomplete braided monoidal category in which the tensor product commutes with colimits and let $F : \mathcal{C} \longrightarrow \mathcal{V}_0$ be a monoidal functor. Then there exists a D-morphism ϕ on the bialgebra $A = \mathrm{coend}(F)$ such that for every object X of $\mathcal{C}$, one has $(1_{F(X)} \otimes \phi) \circ \alpha_X = F(R_X)$.*

Proof. We use the notation and results of Section 2. The D-structure of $\mathcal{C}$ is denoted by R. Then $F(R)$ is an element of $\mathrm{Nat}(F, F)$ and thus there exists an element $\phi \in \mathrm{Hom}_{\mathcal{V}}(A, I)$ such that $\theta_I(\phi) = F(R) = (1_F \otimes \phi) \circ \alpha$, that is for every object X of $\mathcal{C}$, one has $F(R_X) = (1_{F(X)} \otimes \phi) \circ \alpha_X$. It is easily seen that $\phi \circ u = 1_I$ (since $R_I = 1_I$) and that $(\phi \otimes 1_A) \circ \Delta = (1_A \otimes \phi) \circ \Delta$ (since $F(R)$ is a morphism of A-comodules). Now let

$T = \theta_I^2(\psi)$ where $\psi = \phi \circ m$. For all objects X, Y in $\mathcal{C}$ we have

$$\begin{aligned} T_{X,Y} &= (1_{F(X)} \otimes 1_{F(Y)} \otimes \psi) \circ \alpha_{X\otimes Y} \\ &= (1_{F(X)} \otimes 1_{F(Y)} \otimes \phi) \circ \theta_A^2(m)_{X,Y} \\ &= (1_{F(X)} \otimes 1_{F(Y)} \otimes \phi) \circ (\widetilde{F}_{X,Y}^{-1} \otimes 1_A) \circ \alpha_{X\otimes Y} \circ \widetilde{F}_{X,Y} \\ &= \widetilde{F}_{X,Y}^{-1} \circ (1_{F(X\otimes Y)} \otimes \phi) \circ \alpha_{X\otimes Y} \circ \widetilde{F}_{X,Y} \\ &= \widetilde{F}_{X,Y}^{-1} \circ F(R_{X\otimes Y}) \circ \widetilde{F}_{X,Y}. \end{aligned}$$

It is easy to see that

$$(T_{X,Y} \otimes 1_Z) \circ (1_X \otimes T_{Y,Z}) = (1_X \otimes T_{Y,Z}) \circ (T_{X,Y} \otimes 1_Z).$$

for all objects X, Y, Z. This identity is used in Figure 16 to show that

$$\theta_I^3((\psi \otimes \psi) \circ (1_A \otimes \Delta \otimes 1_A))_{X,Y,Z} = \theta_I^3((\psi \otimes \psi) \circ (1_A \otimes \Delta^0 \otimes 1_A))_{X,Y,Z}.$$

This concludes the proof. □

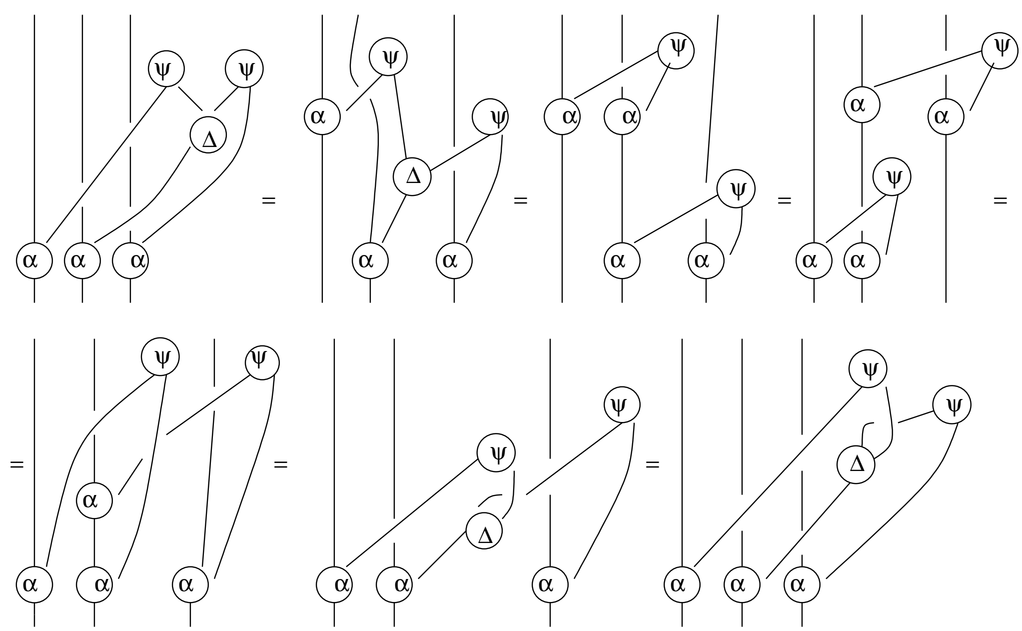


Figure 16: Proof of Proposition 5.5

All the computations have been done to prove:

Corollary 5.6 *Let M be an object having a left dual in a cocomplete braided monoidal category in which the tensor product commutes with colimits, and let $R : M \otimes M \longrightarrow M \otimes M$ be a D-morphism. Then the FRT bialgebra $A(R)$ has a D-morphism $\phi : A(R) \longrightarrow I$ such that $(1_M \otimes \phi) \circ \alpha_M = R$.*

Proof. Let us consider the category $\mathbb{D}$ and the monoidal functor $F_R : \mathbb{D} \longrightarrow \mathcal{V}$ such that $F(1) = M$ and $F(s_1) = R$: $A(R) = \mathrm{coend}(F_R)$. Let us choose a D-structure on $\mathbb{D}$ such that $R_2 = s_1$. Then Proposition 5.5 ensures the existence of the required D-morphism on $A(R)$. □

It is clear that the D-morphism on $A(R)$ is not unique and depends on the choice of a D-structure on $\mathbb{D}$. Let us take a look at the classical example of vector spaces over a field.

Let M be a finite-dimensional k-vector space with basis $(e_i)_{1\leq i\leq n}$ and let $R \in \mathrm{End}_k(M\otimes M)$ with $R(e_i \otimes e_j) = \sum_{k,l} R^{ij}_{kl} e_k \otimes e_l$. The FRT bialgebra $A(R)$ has generators $(x_{ij})_{1\leq i,j\leq n}$ and is submitted to the relations:

$$\sum_{k,l} R^{ij}_{kl} x_{rk} x_{sl} = \sum_{k,l} R^{kl}_{rs} x_{ki} x_{lj} \quad 1 \leq i, j, r, s \leq n.$$

Assume that R is a D-operator and choose the following D structure on the category $\mathbb{D}$: $R_2 = s_1$ and $R_n = 1_n$ if $n \geq 3$. Then the corresponding D-morphism on $A(R)$ may be described as follows. One has:

$$\phi(x_{ij}) = \delta_{ij}, \quad \phi(x_{ki}x_{lj}) = R^{ij}_{kl}, \quad \phi(x_{i_1j_1} \dots x_{i_nj_n}) = \delta_{i_1j_1} \dots \delta_{i_nj_n} \quad \text{if } n \geq 3.$$

If we choose the D-structure such that $R_n = s_1$, $\forall n$, then the corresponding D-morphism on $A(R)$ satisfies:

$$\phi(x_{ij}) = \delta_{ij}, \quad \phi(x_{ki}x_{lj}) = R^{ij}_{kl}, \quad \phi(x_{i_1j_1} \dots x_{i_nj_n}) = R^{j_1j_2}_{i_1i_2} \delta_{i_3j_3} \dots \delta_{i_nj_n} \quad \text{if } n \geq 3.$$

6 Examples

We have already mentioned the fact that certain D-morphisms arise from Frobenius algebras. Let us make this assertion more precise. This is an application of the constructions of section 4.

We still work in a monoidal category $\mathcal{V}$. Let us recall that a Frobenius algebra (A, m, u, t, δ) in $\mathcal{V}$ consists of an algebra (A, m, u) in $\mathcal{V}$, and of morphisms $t : A \longrightarrow I$ and $\delta : I \longrightarrow A \otimes A$ such that letting $e = t \circ m$, one has

$$(1_A \otimes e) \circ (\delta \otimes 1_A) = 1_A \quad \text{and} \quad (e \otimes 1_A) \circ (1_A \otimes \delta) = 1_A.$$

This implies that A has a left dual and is self-dual. It should be noted that the morphism δ is uniquely determined by (A, m, u, t). Given a Frobenius algebra $A = (A, m, u, t, \delta)$ we let $m^* = (m \otimes 1_A) \circ (1_A \otimes \delta) : A \longrightarrow A \otimes A$.

Proposition 6.1 *Let $A = (A, m, u, t, \delta)$ be a Frobenius algebra in $\mathcal{V}$. Then the morphism $R = (1_A \otimes m) \circ (m^* \otimes 1_A)$, $A \otimes A \longrightarrow A \otimes A$, is a solution of the D-equation.*

Proof. Let us show that $\mathcal{L} = (A, A, m^*, m)$ is a Long system:

$$\begin{aligned} m^* \circ m &= (m \otimes 1_A) \circ (1_A \otimes \delta) \circ m \\ &= (m \otimes 1_A) \circ (m \otimes 1_A \otimes 1_A) \circ (1_A \otimes 1_A \otimes \delta) \\ &= (m \otimes 1_A) \circ (1_A \otimes m \otimes 1_A) \circ (1_A \otimes 1_A \otimes \delta) \\ &= (m \otimes 1_A) \circ (1_A \otimes m^*). \end{aligned}$$

Thus $R = R_{\mathcal{L}}$ is a solution of the D-equation by Proposition 4.3. □

Let (Z, m, u, t) be a Frobenius algebra in the category of vector spaces over a field k, and let us consider its quantum automorphism group $A_{aut}(Z, t)$ [21, 2]. It is clear from the proof of Theorem 3.1 in [2] that the D-operator constructed in Proposition 6.1 is a morphism in the category of $A_{aut}(Z, t)$-comodules. Thus there exists a surjective bialgebra morphism $A(R) \longrightarrow A_{aut}(Z, t)$.

Let us now examine some specific examples. Let $A = k^n$ with its standard basis $(e_i)_{1 \le i \le n}$ such that $e_i e_j = \delta_{ij} e_i$ and $\sum_{i=1}^n e_i = 1$. Let t be defined by $t(e_i) = 1$, $\forall i$. Then the operator R of Proposition 6.1 is defined by $R(e_i \otimes e_j) = \delta_{ij} e_i \otimes e_j$. So this operator is a special case of (2), Example 2.4 in [10].

A new example is constructed as follows. Consider the matrix algebra $A = M_n(k)$ with its standard basis $(E_{ij})_{1 \le i,j \le n}$. Let $q \in k^*$ and let $\mathrm{tr}_q : A \longrightarrow k$ be the k-linear map defined by $\mathrm{tr}_q(E_{ij}) = \delta_{ij} q^{i-1}$ (the q-trace). Then it is easily seen that (A, tr_q) is a Frobenius algebra, and Proposition 6.1 furnishes the D-operator

$$R(E_{ij} \otimes E_{kl}) = \delta_{jk} \sum_{r=1}^{n} q^{1-r} E_{ir} \otimes E_{rl}.$$

We now wish to get solutions of the D-equation using finite groups. For this we use the results of section 5. Let G be a finite group and let $\mathcal{F}(G)$ be the Hopf algebra of k-valued functions on G.

Definition 6.2 *A function $\beta : G \longrightarrow k$ is said to be a D-function if it satisfies the following three conditions:*
a) $\sum_{x \in G} \beta(x) = 1$;
b) β is constant on the conjugacy classes of G ;
c) If $x, y \in G$ are such that $xy \neq yx$, then $\beta(x) = 0$ or $\beta(y) = 0$.

The proof of the following easy result is left to the reader:

Proposition 6.3 *Let $\beta : G \longrightarrow k$ be a D-function. Then the linear map $\phi_\beta : \mathcal{F}(G) \longrightarrow k$, $f \longmapsto \sum_{x \in G} \beta(x) f(x)$, is a D-morphism on the Hopf algebra $\mathcal{F}(G)$. Conversely, any D-morphism on $\mathcal{F}(G)$ is of the form ϕ_β for a unique D-function $\beta : G \longrightarrow k$.* □

Let us note that if G is an abelian group any function on G satisfying condition a) of definition 6.2 is a D-function and thus any linear form ϕ on $\mathcal{F}(G)$ such that $\phi(1) = 1$ is a D-morphism. More generally it is obvious that any linear form ϕ on a cocommutative bialgebra such that $\phi(1) = 1$ is a D-morphism. Let us now compute some specific examples.

• Let $G = \langle x \mid x^2 = 1 \rangle$ be the cyclic group of order 2. Let $q \in k$ and let $\beta : G \longrightarrow k$ be the D-function defined by $\beta(1) = 1 - q$ and $\beta(x) = q$. Let $M = \mathcal{F}(G)$ with its δ-basis δ_1, δ_x.

Then the D-morphism $R : M \otimes M \longrightarrow M \otimes M$ furnished by Proposition 5.4, with respect to the ordered basis $\{\delta_1 \otimes \delta_1, \delta_1 \otimes \delta_x, \delta_x \otimes \delta_1, \delta_x \otimes \delta_x\}$, is given by:

$$R = \begin{pmatrix} 1-q & 0 & 0 & q \\ 0 & 1-q & q & 0 \\ 0 & q & 1-q & 0 \\ q & 0 & 0 & 1-q \end{pmatrix}.$$

• This preceding example is a special case of the following one. Let $G = \langle x \mid x^n = 1 \rangle$ be the cyclic group of order n. Let $q_0, q_1, \ldots, q_{n-1} \in k$ with $\sum_{i=0}^{n-1} q_i = 1$ and let $\beta : G \longrightarrow k$ be the D-function defined by $\beta(x^i) = q_i$. Let $M = \mathcal{F}(G)$ with its δ-basis $\delta_1, \delta_x, \ldots, \delta_{x^{n-1}}$. We get the following D-operator:

$$R(\delta_{x^i} \otimes \delta_{x^j}) = \sum_{k=0}^{n-1} \beta(x^{i-k}) \delta_{x^k} \otimes \delta_{x^{k+j-i}}.$$

• Let G be the symmetric group S_3. Let $q \in k$ and let $\beta : G \longrightarrow k$ be the unique D-function defined by

$$\beta(1) = 1 - 2q \ , \quad \beta((1,2,3)) = q \ , \quad \beta((1,2)) = 0.$$

Consider the 2-dimensional irreducible representation $\pi : S_3 \longrightarrow \mathrm{GL}_2(k)$ defined by

$$\pi((1,2)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad ; \quad \pi((1,2,3)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Let $M = \mathbb{C}^2$. The D-operator R on M furnished by Proposition 5.4, in the standard ordered basis, is given by:

$$R = \begin{pmatrix} 1-q & -q & -q & 2q \\ q & 1-2q & -2q & q \\ q & -2q & 1-2q & q \\ 2q & -q & -q & 1-q \end{pmatrix}.$$

Acknowledgements. Part of this work was done during a visit of the first author to Macquarie University. He wishes to thank the Australian Research Council and the Laboratory G.T.A (University of Montpellier II, UMR 5030) for their financial support. Special thanks to G. Laffaille and C. Cibils for suggesting the visit.

References

[1] J. Benabou, Introduction to bicategories, Lecture Notes in Math. 47, 1-77, Springer-Verlag, 1967.

[2] J. Bichon, Galois reconstruction of finite quantum groups, J. Algebra 230, 683-693, 2000.

[3] A. Joyal, R. Street, An introduction to Tannaka duality and quantum groups, Lecture Notes in Math. 1488, 413–492, Springer-Verlag, 1991.

[4] A. Joyal, R. Street, The geometry of tensor calculus, I, Advances in Math. 88, 55-112, 1991.

[5] A. Joyal, R. Street, Braided tensor categories, Advances in Math. 102, 20-73, 1993.

[6] C. Kassel, Quantum Groups, GTM 155, Springer, 1995.

[7] R.G. Larson, J. Towber, Two dual classes of bialgebras related to the concept of "quantum group" and "quantum Lie algebra", Comm. Algebra 19, 3295-3245, 1991.

[8] S. Mac Lane, Categories for the working mathematician, GTM 5, Springer, 1971.

[9] P. McCrudden, Categories of representations of coalgebroids, Thesis, Macquarie University, 1999.

[10] G. Militaru, The Long dimodules category and non-linear equations, Algebr. Represent. Theory 2, 177-200, 1999.

[11] G. Militaru, A class of non-symmetric solutions for the integrability condition of the Knizhnik-Zamolodchikov equation: a Hopf algebra approach, Comm. Algebra 27(5), 2393-2407, 1999.

[12] B. Pareigis, Non additive ring and module theory II. $\mathcal{C}$-categories, $\mathcal{C}$-functors and $\mathcal{C}$-morphisms. Publ. Math Debrecen 25, 351-361, 1977.

[13] B. Pareigis, Reconstruction of hidden symmetries, J. Algebra 183, 90-154, 1996.

[14] D. Radford, Solutions to the quantum Yang-Baxter equation and the Drinfeld double, J. Algebra 161, 20-32, 1993.

[15] N. Yu Reshethikin, L.A. Takhtajan, L.D. Faddeev, Quantizations of Lie groups and Lie algebras, Leningrad Math. J. 1, 1990, 193-225.

[16] N. Saavedra Rivano, Catégories tannakiennes, Lecture Notes in Math. 265, Springer-Verlag, 1972.

[17] P. Schauenburg, Tannaka duality for arbitrary Hopf algebras, Algebra Berichte 66, Verlag Reinhard Fischer, Munich, 1992.

[18] R. Street, Quantum Groups: An entrée to modern algebra, 1989, available at www-texdev.mpce.mq.edu.au/Quantum/.

[19] M.E. Sweedler, Hopf algebras, Benjamin, 1969.

[20] K.H. Ulbrich, On Hopf algebras and rigid monoidal categories, Israel J. Math. 72, 252–256, 1990.

[21] S. Wang, Quantum symmetry groups of finite spaces, Comm. Math. Phys. 195, 1998, 195-211.

Julien Bichon
Laboratoire de Mathématiques Appliquées
Université de Pau et des Pays de l'Adour
IPRA, Avenue de l'université
64000 Pau, France
E-mail : Julien.Bichon@univ-pau.fr

Ross Street
Department of Mathematics
Macquarie University
Sydney, NSW 2109, Australia
E-mail : street@ics.mq.edu.au