Monoidal invariance of the cohomological dimension of Hopf algebras: the finite case

Julien Bichon
Université Clermont Auvergne

CroCQG 2026, Besançon

The question I want to discuss is

Question A, 2016

Let A, B be Hopf algebras such that

$$\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$$

Do we have $\mathrm{gldim}(A) = \mathrm{gldim}(B)$?

Here:

- $\mathcal{M}^A$ is the tensor category of right A-comodules,
- $\mathrm{gldim}(A)$ is the global dimension of A (see below).

We work over $\mathbb{C}$.

Global dimension

Let A be an algebra and let M be a (left) A-module.

• M is said to be **projective** if the functor $\mathrm{Hom}_A(M, -)$ is exact. This is equivalent to say that M is a direct summand in a free module.

• A **projective resolution** of M is an exact sequence of A-modules

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

where the P_i's are projective.

• The **projective dimension of** M, $\mathrm{pd}_A(M) \in \mathbb{N} \cup \{\infty\}$, is the smallest possible length (the largest n with $P_n \neq 0$) for a projective resolution of M.

• We have $\mathrm{pd}_A(M) = 0 \iff M$ is projective, so $\mathrm{pd}_A(M)$ measures how far is a module from being projective.

• The **(left) global dimension of** A is defined by

$$\mathrm{gldim}(A) = \max \{\mathrm{pd}_A(M),\ M \in {}_A\mathcal{M}\} \in \mathbb{N} \cup \{\infty\}$$

Global dimension

More generally as soon as we are in an abelian category having enough projective objects (every object is a quotient of a projective), we can define projective dimensions of objects.

When A is a Hopf algebra, we have

$$\mathrm{gldim}(A) = \mathrm{pd}_A({}_\varepsilon k) = \mathrm{cd}(A)$$

where ${}_\varepsilon k$ denotes the trivial A-module, and

$\mathrm{cd}(A)$ is the **Hochschild cohomological dimension of** A

with $\mathrm{cd}(A) = \mathrm{pd}_{{}_A\mathcal{M}_A}(A)$.

In this situation we simply say that $\mathrm{cd}(A) = \mathrm{gldim}(A)$ is the cohomological dimension of A.

Remark: we could also define the global dimension by looking at right modules, this would be the right global dimension, but in the Hopf algebra case this always coincides with the left one.

Classical examples

- For $X \subset \mathbb{C}^n$ an affine algebraic set, X is smooth $\iff \mathrm{gldim}(\mathcal{O}(X))$ is finite, and in this case if X is connected, $\dim(X) = \mathrm{gldim}(\mathcal{O}(X))$.
- If $A = \mathcal{O}(G)$, with G a compact Lie group, then
$$\mathrm{gldim}(\mathcal{O}(G)) = \dim(G)$$
- $\mathrm{gldim}(\mathbb{C}[X]/(x^2)) = \infty$.
- For Γ a discrete group, we have
  - $\mathrm{gldim}(\mathbb{C}\Gamma) = 0 \iff |G|$ is finite.
  - if Γ is finitely generated, then $\mathrm{gldim}(\mathbb{C}\Gamma) = 1 \iff \Gamma$ has a free subgroup of finite index (Dunwoody's theorem);
  - if Γ is the fundamental group of an aspherical closed manifold of dimension n, then $\mathrm{gldim}(\mathbb{C}\Gamma) = n$.
  - Let $\Gamma = \langle a, s, t \mid st = ts, tat^{-1}a = atat^{-1}, sas^{-1} = atat^{-1} \rangle$ (Baumslag). Then $\mathrm{gldim}(\mathbb{C}\Gamma) = \infty$.
- If A is a finite-dimensional Hopf algebra, then either $\mathrm{gldim}(A) = 0$ (A is semisimple) or $\mathrm{gldim}(A) = \infty$.

Global dimension and monoidal equivalences: examples

Example Let $n \geq 2$. Let $A_o(n)$ be the algebra presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and relations

$$u^t u = I_n = uu^t$$

where u is the matrix $(u_{ij})_{1\leq i,j\leq n}$. It has a Hopf algebra structure

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},\ \varepsilon(u_{ij}) = \delta_{ij},\ S(u) = u^t$$

This is the coordinate algebra on Wang's free orthogonal quantum group O_n^+. Collins-Härtl-Thom (2008) have shown

$$\mathrm{gldim}(A_o(n)) = 3$$

There is a monoidal equivalence $\mathcal{M}^{A_o(n)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}(SL_q(2))}$ for $n = -q - q^{-1}$, and indeed $\mathrm{gldim}(\mathcal{O}(SL_q(2))) = 3$.

Global dimension: examples

Example (continued). More generally, let $E \in \mathrm{GL}_n(\mathbb{C})$, $n \geq 2$, and let $\mathcal{B}(E)$ presented by generators $(u_{ij})_{1 \leq i,j \leq n}$ and relations

$$E^{-1}u^tEu = I_n = uE^{-1}u^tE,$$

where u is the matrix $(u_{ij})_{1 \leq i,j \leq n}$. It has a Hopf algebra structure defined by $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj},\ \varepsilon(u_{ij}) = \delta_{ij},\ S(u) = E^{-1}u^tE$.
The Hopf algebra $\mathcal{B}(E)$ (Dubois-Violette and Launer, 1990), represents the quantum symmetry group of the bilinear form associated to the matrix E. For a well-chosen $E_q \in \mathrm{GL}_2(k)$ we have $\mathcal{B}(E_q) = \mathcal{O}(\mathrm{SL}_q(2))$.
One has

$$\mathrm{gldim}(\mathcal{B}(E)) = 3$$

and we have a monoidal equivalence

$$\mathcal{M}^{\mathcal{B}(E)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}(\mathrm{SL}_q(2))}$$

for $q \in \mathbb{C}^*$ satisfying $\mathrm{tr}(E^{-1}E^t) = -q - q^{-1}$.

Global dimension: examples

Example. Let $A_s(n)$ be the algebra presented by generators $(u_{ij})_{1 \leq i,j \leq n}$ and relations

$$\sum_k u_{ki} = 1 = \sum_k u_{ik},\ u_{ik}u_{ij} = \delta_{kj}u_{ij},\ u_{ki}u_{ji} = \delta_{jk}u_{ji}$$

It has a natural Hopf algebra structure and represents the quantum permutation group S_n^+ (Wang).
For $n \geq 4$, one has

$$\mathrm{gldim}(A_s(n)) = 3$$

and a monoidal equivalence $\mathcal{M}^{A_s(n)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}(PSL_q(2))}$ for $\sqrt{n} = q + q^{-1}$.

In these examples, the monoidal equivalence is important to determine the cohomological dimension, but there are furthermore special types of "equivariant" resolutions that play a role.

Global dimension: examples

Example. Let $n \geq 2$, and let $A_u(n)$ be the $*$-algebra presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and relations

$$u^* u = I_n = uu^* = \overline{u}u = u^t\overline{u},$$

corresponding to the free unitary quantum group U_n^+. We have

$$\mathrm{gldim}(A_u(n)) = 3$$

Example. $\mathrm{gldim}(\mathcal{O}_q(G)) = \dim(G)$ (Levasseur-Stafford for SL_n, Brown-Goodearl, Yakimov...)

Example. $\mathrm{gldim}(U_q(\mathfrak{g})) = \dim(\mathfrak{g})$ (Brown-Goodearl, Chemla...)

Global dimension: examples

Example. For $p, n \geq 1$, let $A_h^p(n)$ be presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and relations

$$\sum_k u_{ki}^p = 1 = \sum_k u_{ik}^p, \ \ u_{ik}u_{ij} = 0 = u_{ki}u_{ji} \text{ for } k \neq j$$

It has a natural Hopf algebra structure ($S(u_{ij}) = u_{ji}^{p-1}$) and corresponds to a so-called quantum reflection group. For $n \geq 4$, we have

$$\mathrm{gldim}(A_h^p(n)) = 3$$

This follows from the formula

$$\mathrm{gldim}(A *_w A_s(n)) = \max(\mathrm{gldim}(A), 3)$$

for any compact Hopf algebra A and any $n \geq 4$, where $*_w$ is the free wreath product ($A_h^p(n) \simeq \mathbb{C}[\mathbb{Z}/p\mathbb{Z}] *_w A_s(n)$).

R. Zhu's counterexample to Question A

After a number of positive partial answers to Question A, R. Zhu (2025) produced a counterexample. Indeed the Hopf algebras

$$A = \mathbb{C}\langle x, g^{\pm 1} \mid x^2 = 0, gx = -xg\rangle,$$
$$B = \mathbb{C}\langle x, g^{\pm 1} \mid x^2 = (1 - g^2), gx = -xg\rangle,$$

with

$$\Delta(x) = 1 \otimes x + x \otimes g,\ \Delta(g) = g \otimes g$$

satisfy

$$\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B,\ \mathrm{gldim}(A) = \infty,\ \mathrm{gldim}(B) = 1$$

A Partial Positive answer to Question A

Between 2016 and 2025, there have been a number of partial positive answers to Question A.
A more recent work of Ren and Zhu (2026) on Gorenstein projective dimensions has the following consequence:

Theorem

Let A, B be Hopf algebras with bijective antipodes and with $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. If $\mathrm{gldim}(A)$ and $\mathrm{gldim}(B)$ are finite, then $\mathrm{gldim}(A) = \mathrm{gldim}(B)$.

In this talk I wish to explain a slightly simpler appraoch to this result, that also has the merit to remove the assumption on the bijectivity of the antipodes.

Strategy

Let A, B be Hopf algebras such that $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$.
It turns out (Ulbrich, Schauenburg) that such a monoidal equivalence always arise (via cotensor product) from an algebra R which an A-B-bi-Galois object :

$$\begin{aligned} \mathcal{M}^A &\longrightarrow \mathcal{M}^B \\ V &\longmapsto V \square_A R \end{aligned}$$

The strategy is to compare $\mathrm{gldim}(A)$ and $\mathrm{gldim}(B)$ via $\mathrm{cd}(R)$ (and not $\mathrm{gldim}(R)$!) by showing that $\mathrm{gldim}(A) = \mathrm{cd}(R) = \mathrm{gldim}(B)$.

Why not $\mathrm{gldim}(R)$: this is ruled out by the example with $A = \mathbb{C}\mathbb{Z}^2 = B$, $R = \mathcal{O}_q(\mathbb{T}^2) = \mathbb{C}\langle x^{\pm 1}, y^{\pm 1}\rangle/(yx = qxy)$, with q not a root of unity, for which $\mathrm{gldim}(R) = 1 < \mathrm{cd}(R) = 2 = \mathrm{gldim}(A) = \mathrm{gldim}(B) = 2$.

Canonical functors

Let A be a Hopf algebra and let R be a left (resp. right) A-comodule algebra. Recall that the *canonical functor associated to* R is the functor

$$\begin{aligned} F_R^l : {}_A\mathcal{M} &\longrightarrow {}_R\mathcal{M}_R \\ V &\longmapsto V \odot R \end{aligned} \qquad \left(\text{resp. } \begin{aligned} F_R^r : \mathcal{M}_A &\longrightarrow {}_R\mathcal{M}_R \\ V &\longmapsto R \odot V \end{aligned}\right)$$

where $V \odot R$ (resp $R \odot V$) is $V \otimes R$ (resp. $R \otimes V$) as a vector space, and has R-bimodule structure given by

$$x \cdot (v \otimes y) \cdot z = x_{(-1)} \cdot v \otimes x_{(0)}yz, \quad x, y, z \in R, \ v \in V$$
$$(\text{resp. } x \cdot (y \otimes v) \cdot z = xyz_{(0)} \otimes v \cdot z_{(1)} \quad x, y, z \in R, \ v \in V)$$

We have $F_R^l({}_\varepsilon\mathbb{C}) = R = F_R^r(\mathbb{C}_\varepsilon)$ so **these functors possibly connect the (left,right) global dimension of A with the cohomological dimension of** R, if they preserve projective dimensions (which in general does not happen).

The key technical result

If $\mathcal{S}$ is a class of objects of an abelian category $\mathcal{C}$, the notation $\mathrm{add}(\mathcal{S})$ stands for the full subcategory of $\mathcal{C}$ whose objects are the finite direct sums of direct summands of objects of $\mathcal{S}$. The notation $\mathrm{Proj}(\mathcal{C})$ stands for the class of projective objects of $\mathcal{C}$.
The following result is a variation/simplification on a result of Chen-Ren (2022):

Proposition

Let $\mathcal{C}, \mathcal{D}$ be abelian categories having enough projective objects, and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors. Assume that F and G are exact and that $\mathrm{add}(G(\mathrm{Proj}(\mathcal{D}))) = \mathrm{Proj}(\mathcal{C})$. Then, for any object X in $\mathcal{C}$ such that $\mathrm{pd}_{\mathcal{C}}(X)$ is finite, we have $\mathrm{pd}_{\mathcal{C}}(X) = \mathrm{pd}_{\mathcal{D}}(F(X))$.

The key technical result

Proposition

Let $\mathcal{C}, \mathcal{D}$ be abelian categories having enough projective objects, and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors. Assume that F and G are exact and that $\mathrm{add}(G(\mathrm{Proj}(\mathcal{D}))) = \mathrm{Proj}(\mathcal{C})$. Then, for any object X in $\mathcal{C}$ such that $\mathrm{pd}_{\mathcal{C}}(X)$ is finite, we have $\mathrm{pd}_{\mathcal{C}}(X) = \mathrm{pd}_{\mathcal{D}}(F(X))$.

Proof: G exact and F left adjoint to $G \Rightarrow F$ preserves projective objects, and then F exact $\Rightarrow$ for any $X \in \mathrm{ob}(\mathcal{C})$, $\mathrm{pd}_{\mathcal{D}}(F(X)) \leq \mathrm{pd}_{\mathcal{C}}(X)$. Notice also that for $Q \in \mathrm{ob}(\mathcal{D})$

$$\mathrm{Ext}^*_{\mathcal{D}}(F(X), Q) \simeq \mathrm{Ext}^*_{\mathcal{C}}(X, G(Q))$$

which also proves that $\mathrm{pd}_{\mathcal{D}}(F(X)) \leq \mathrm{pd}_{\mathcal{C}}(X)$. Assume $\mathrm{pd}_{\mathcal{C}}(X) = m$ is finite. Long $\mathrm{Ext}^*_{\mathcal{C}}$ exact sequence $\rightsquigarrow$ projective object P in $\mathcal{C}$ with $\mathrm{Ext}^m_{\mathcal{C}}(X, P) \neq \{0\}$. Last assumption $\rightsquigarrow$ $Q \in \mathrm{Proj}(\mathcal{D})$ and $P' \in \mathrm{ob}(\mathcal{C})$ with $G(Q) \simeq P \oplus P'$. Hence

$$\mathrm{Ext}^m_{\mathcal{D}}(F(X), Q) \simeq \mathrm{Ext}^m_{\mathcal{C}}(X, G(Q)) \simeq \mathrm{Ext}^m_{\mathcal{C}}(X, P \oplus P') \simeq \mathrm{Ext}^m_{\mathcal{C}}(X, P) \oplus \mathrm{Ext}^m_{\mathcal{C}}(X, P')$$

$\rightsquigarrow$ $\mathrm{pd}_{\mathcal{D}}(F(X)) \geq m = \mathrm{pd}_{\mathcal{C}}(X)$, which finishes the proof. $\square$

Cogroupoids

Recall (Hoa's talk) that a cogroupoid $\mathbf{C}$ consists of a set of objects $\mathrm{ob}(\mathbf{C})$, with for any objects X, Y, algebras $\mathbf{C}(X, Y)$, together with algebra maps

$$\Delta_{XY}^{Z} : \mathbf{C}(X, Y) \to \mathbf{C}(X, Z) \otimes \mathbf{C}(Z, Y)$$

$$\varepsilon_X : \mathbf{C}(X, X) \to \mathbb{C}$$

$$S_{XY} : \mathbf{C}(X, Y) \to \mathbf{C}(Y, X)$$

satisfying axioms that are dual to that of a groupoid.

It follows in particular that each $\mathbf{C}(X, X)$ is a Hopf algebra and each $\mathbf{C}(X, Y)$ is a $\mathbf{C}(X, X)$-$\mathbf{C}(Y, Y)$ bicomodule algebra.

A connected cogroupoid is said to be connected if $\mathbf{C}(X, Y)$ is non-zero for any objects X, Y.

Cogroupoids

Proposition

Let $\mathbf{C}$ be a connected cogroupoid. For $X, Y \in \mathrm{ob}(\mathbf{C})$, the canonical functors

$$F^l_{\mathbf{C}(X,Y)} : {}_{\mathbf{C}(X,X)}\mathcal{M} \longrightarrow {}_{\mathbf{C}(X,Y)}\mathcal{M}_{\mathbf{C}(X,Y)},$$

$$F^r_{\mathbf{C}(X,Y)} : \mathcal{M}_{\mathbf{C}(Y,Y)} \longrightarrow {}_{\mathbf{C}(X,Y)}\mathcal{M}_{\mathbf{C}(X,Y)}$$

satisfy the assumption of the key technical result.

Proof: The corresponding right adjoint functors come from the following algebra maps:

$$\mathbf{C}(X,X) \to \mathbf{C}(X,Y) \otimes \mathbf{C}(X,Y)^{\mathrm{op}}, \ a^{XX} \mapsto a^{XY}_{(1)} \otimes S_{YX}(a^{YX}_{(2)})$$

$$\mathbf{C}(Y,Y) \to \mathbf{C}(X,Y)^{\mathrm{op}} \otimes \mathbf{C}(X,Y), \ a^{YY} \mapsto S_{YX}(a^{YX}_{(1)}) \otimes a^{XY}_{(2)}$$

The main point is then to show that the corresponding functors transform free bimodules into free modules.

For example, for the second one,

$$\kappa_r^* : {}_{\mathbf{C}(X,Y)}\mathcal{M}_{\mathbf{C}(X,Y)} \longrightarrow \mathcal{M}_{\mathbf{C}(Y,Y)}$$

this follows from the fact that the above right $\mathbf{C}(Y,Y)$-module map

$$\begin{aligned} \mathbf{C}(X,Y) \otimes \mathbf{C}(Y,Y)_{\mathbf{C}(Y,Y)} &\longrightarrow \kappa_r^*(\mathbf{C}(X,Y) \otimes \mathbf{C}(X,Y)) \\ a^{XY} \otimes b^{YY} &\longmapsto S_{YX}(b^{YX}_{(1)})a^{XY} \otimes b^{XY}_{(2)} \end{aligned}$$

is an isomorphism with inverse

$$\begin{aligned} \mathbf{C}(X,Y) \otimes \mathbf{C}(X,Y) &\longrightarrow \mathbf{C}(X,Y) \otimes \mathbf{C}(Y,Y) \\ a^{XY} \otimes b^{XY} &\longmapsto S_{YX}S_{XY}(b^{XY}_{(1)})a^{XY} \otimes b^{YY}_{(2)} \quad \square \end{aligned}$$

Cogroupoids

Proposition

Let $\mathbf{C}$ be a connected cogroupoid. For $X, Y \in \mathrm{ob}(\mathbf{C})$, the canonical functors

$$F^l_{\mathbf{C}(X,Y)} : {}_{\mathbf{C}(X,X)}\mathcal{M} \longrightarrow {}_{\mathbf{C}(X,Y)}\mathcal{M}_{\mathbf{C}(X,Y)},$$

$$F^r_{\mathbf{C}(X,Y)} : \mathcal{M}_{\mathbf{C}(Y,Y)} \longrightarrow {}_{\mathbf{C}(X,Y)}\mathcal{M}_{\mathbf{C}(X,Y)}$$

satisfy the assumption of the key technical result.

We thus obtain, in a connected cogroupoid, if $\mathrm{gldim}(\mathbf{C}(X,X))$ and $\mathrm{gldim}(\mathbf{C}(Y,Y))$ are finite,

$$\mathrm{gldim}(\mathbf{C}(X,X)) = \mathrm{cd}(\mathbf{C}(X,Y)) = \mathrm{gldim}(\mathbf{C}(Y,Y))$$

Cogroupoids

We thus obtain:

Theorem

Let A, B be Hopf algebras with $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. If $\mathrm{gldim}(A)$ and $\mathrm{gldim}(B)$ are finite, then $\mathrm{gldim}(A) = \mathrm{gldim}(B)$.

Indeed, the Ulbrich-Schauenburg corresponding Hopf-Galois object can be given even more structure: there is a connected cogroupoid $\mathbf{C}$ such that $\mathbf{C}(X, X) = A$ and $\mathbf{C}(Y, Y) = B$, so we can us the result in the previous slide to conclude $\mathrm{gldim}(A) = \mathrm{gldim}(B)$.

What do we do now?

Let A, B be Hopf algebras with $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$. We have seen that

1. it is possible that $\mathrm{gldim}(A) = 1$ and $\mathrm{gldim}(B) = \infty$, so that Question A has a negative answer;
2. if $\mathrm{gldim}(A)$ and $\mathrm{gldim}(B)$ are finite, we have $\mathrm{gldim}(A) = \mathrm{gldim}(B)$.

This seems to be optimal, but we still can wonder whether if, by restricting to some special classes of Hopf algebras, one can get a positive answer to Question A. More precisely, we have the following new question:

Question B

Let A, B be cosemimple Hopf algebras such that

$$\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$$

Do we have $\mathrm{gldim}(A) = \mathrm{gldim}(B)$?

What do we do now?

Question B

Let A, B be cosemimple Hopf algebras such that

$$\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$$

Do we have $\mathrm{gldim}(A) = \mathrm{gldim}(B)$?

The following result supports a positive answer:

Theorem, B 2016-2018

Let A, B be cosemimple Hopf algebras with $S^4 = \mathrm{id}$. If $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ then $\mathrm{gldim}(A) = \mathrm{gldim}(B)$.

Remark: in Zhu's counterexample we have given, the antipodes satisfy $S^4 = \mathrm{id}$.