SOME CLASSICAL SPACES REALIZED AS MUTATIONAL SPACES

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Abstract. We show that several classical spaces, including compact Lie groups, Euclidean spheres and projective spaces, admit a natural mutational space structure.

1. Introduction

The theory of mutational calculus, introduced by J.P. Aubin [1], is a generalization of the ordinary differential calculus on normed spaces to the case of metric spaces. It seems that this theory is not very well-known amongst pure mathematicians. There might be two main reasons for this. First although Aubin's definition of a mutational space only involves basic mathematical notions, it is also quite tricky and it is difficult at first sight to see what is really going on. Second it seems that there is a lack of simple non completely trivial examples, and hence there is no comparison between Aubin's approach and the more traditional differential calculus on manifolds.

It is shown here that that several classical spaces, including compact Lie groups, Euclidean spheres and projective spaces, admit a natural mutational space structure. We hope that these simple examples will be useful experimentation tools for those wishing to prove theorems in mutational analysis.

This work is organized as follows. In Section 2 we recall the definition of a mutational space. In Section 3 we introduce the concept of a quasi-exponential map, modeled on the exponential map of Lie groups, and show that a metric group with a left and right invariant metric and having a quasi-exponential map admits a natural mutational structure. This is applied to compact Lie groups. Unfortunately the left and right invariance condition for the metric restricts our construction to compact groups. Section 4 is devoted to a slight generalization to homogeneous spaces over compact Lie groups, which produces a mutational structure on an Euclidean sphere. In Section 5, we construct a mutational structure on the orbit space of a mutational space for a nice action of a group, called a mutational action. This endows projective spaces with a mutational structure, and gives many examples.

2. Mutational spaces

Let us first recall the very basic definitions around Aubin's mutational calculus [1, 2].

Definition 1. *Let $X = (X, d)$ be a metric space. A* ***transition*** *on X is a continuous map $u : X \times [0, 1] \longrightarrow X$ satisfying the following conditions.*
i) $\forall x \in X,\ u(x, 0) = x$.
ii) $\forall x \in X,\ \forall t \in [0, 1[,\ \lim_{h\to 0^+} \frac{d(u(x,t+h),u(u(x,t),h))}{h} = 0$.

iii) $\alpha(u) = \max(0, \sup_{x\neq y} \limsup_{h\to 0^+} \frac{d(u(x,h),u(y,h))-d(x,y)}{hd(x,y)}) < +\infty$.
iv) $\beta(u) = \sup_{x\in X} \limsup_{h\to 0^+} \frac{d(u(x,h),x)}{h} < +\infty$

The space of transitions on X is denoted by $\mathcal{U}(X)$ The space $\mathcal{U}(X)$ carries a natural metric, defined by

$$d_\Delta(u,v) = \sup_{x\in X} \limsup_{h\to 0^+} \frac{d(u(x,h),v(x,h))}{h}.$$

It is not so immediate that d_Δ is a distance: see [6] for a careful proof.

Definition 2. *A* ***mutational space*** *consists of a pair $(X,\mathcal{D})$ where X is a metric space and $\mathcal{D}$ is a closed subset of $\mathcal{U}(X)$ containing the neutral transition* **1** *defined by* $\mathbf{1}(x,h) = x$, $\forall x \in X$, $\forall h \in [0,1]$.

The basic example is the following one. Let X be a normed space. For $u \in X$, define a transition $\widetilde{u}$ by $\widetilde{u}(x,h) = x + uh$, and let $\mathcal{D}_X = \{\widetilde{u},\ u \in X\}$. Then $(X,\mathcal{D}_X)$ is a mutational space.

3. Quasi-exponential maps and mutational structure on a compact Lie group

Our first goal is to construct a mutational structure on a compact Lie group. Since the differential manifold structure of such an object is determined by the exponential map, it is natural to try to construct a mutational space structure using the exponential map. This leads to the following definition:

Definition 3. *Let G be a metric group. A* ***quasi-exponential map*** *consists of a continuous map $q : V \longrightarrow G$, where V is a (real) Banach space, satisfying the following conditions for $t, h \in \mathbb{R}$ and $a, b \in V$.*
i) $q(ta + ha) = q(ta)q(ha)$.
ii) $\limsup_{h\to 0^+} \frac{d(q(ha),q(hb))}{h} = ||a - b||$.

By a metric group we mean a topological group endowed with a metric compatible with its topology. Definition 3 is motivated by the following classical situation. Let G be a closed subgroup of the general linear group $GL_n(\mathbb{R})$. Let $\mathfrak{g}$ be the Lie algebra of G, which may be described in the following way:

$$\mathfrak{g} = \{a \in M_n(\mathbb{R}) \mid \exp(ta) \in G,\ \forall t \in \mathbb{R}\}.$$

Then $\mathfrak{g}$ is linear subspace of $M_n(\mathbb{R})$ (see e.g. [5]), and the exponential map

$$\exp : \mathfrak{g} \longrightarrow G$$

is a quasi-exponential map. For example, for $G = GL_n(\mathbb{R})$ we have $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$, and for $G = O(n)$ or $G = SO(n)$, we have $\mathfrak{g} = \mathfrak{so}(n) = \{A \in M_n(\mathbb{R}) \mid {}^tA = -A\}$.

We now try to construct transitions using quasi-exponential maps. Recall that a metric d on a group G is said to be left invariant if $d(gx, gy) = d(x, y)$ for all $g, x, y \in G$, and is said to be right invariant if $d(xg, yg) = d(x, y)$ for all $g, x, y \in G$.

Proposition 4. *Let $G = (G, d)$ be a metric group and let $q : V \longrightarrow G$ be a quasi-exponential map. For $a \in V$, define a map*

$$u_a : G\times[0,1] \longrightarrow G$$
$$(x,h) \longmapsto q(ha)x$$

Then u_a is a continuous map satisfying to axioms i) and ii) in Definition 1. If the metric d is left invariant, then u_a satisfies to axiom iii) in Definition 1, with $\alpha(u_a) = 0$. If the metric d is right invariant, then u_a satisfies to axiom iv) in Definition 1, with $\beta(u_a) = ||a||$. Hence if d is a left and right invariant metric on G, then u_a is a transition on G.

Proof. It is clear that u_a is a continuous map, and that axioms i) and ii) in Definition 1 are fulfilled, thanks to axiom i) in Definition 3. Also it is immediate that if d is left invariant, then axiom iii) of Definition 1 is satisfied with $\alpha(u_a) = 0$. Finally, using axiom ii) of Definition 3, the right invariance of the metric d implies that u_a satisfies to axiom iv) in Definition 3 with $\beta(u_a) = ||a||$. □

Corollary 5. *Let G be a metric group endowed with a left and right invariant metric and let $q : V \longrightarrow G$ be a quasi-exponential map. Put $\mathcal{D}_{G,q} = \{u_a,\ a \in V\}$. Then $(G, \mathcal{D}_{G,q})$ is a mutational space, and the metric spaces $\mathcal{D}_{G,q}$ and V are isometric.*

Proof. We have already seen that $\mathcal{D}_{G,q} \subset \mathcal{U}(G)$. Using the right invariance of the metric d and the second axiom of quasi-exponential maps, it is immediate that the map $u : V \longrightarrow \mathcal{U}(G)$ defined by $u(a) = u_a$ is an isometry. Thus since V is a Banach space, $D_{G,q} = u(V)$ is a complete metric space, and we conclude that $\mathcal{D}_{G,q}$ is closed in $\mathcal{U}(G)$. □

It is now easy to construct a mutational space structure on a compact Lie group. Recall that any compact Lie group admits a faithful real representation into an orthogonal group (see e.g. [3]). Thus we just have to treat the case of a closed subgroup of an orthogonal group. We endow the matrix algebra $M_n(\mathbb{R})$ with the operator norm associated with the Euclidean norm on $\mathbb{R}^n$. The induced metric on the orthogonal group $O(n)$ is left and right invariant. Rewriting Proposition 4 and Corollary 5 in this particular case, we have:

Proposition 6. *Let G be a closed subgroup of the orthogonal group $O(n)$, and let $\mathfrak{g}$ be its Lie algebra. For $a \in \mathfrak{g}$, define*

$$\begin{aligned} u_a : G\times[0,1] &\longrightarrow G \\ (x,h) &\longmapsto \exp(ha)x \end{aligned}$$

Then u_a is a transition on G and $(G, \mathcal{D}_G)$ is a mutational space, with $\mathcal{D}_G = \{u_a,\ a \in \mathfrak{g}\}$. □

It is disappointing to restrict to compact groups in Proposition 6, but we have not been able to be more general. Let us examine the situation $G = GL_n(\mathbb{R})$. Endow G with the metric induced by any operator norm on $M_n(\mathbb{R})$. This metric is not left or right invariant. For $a \in M_n(\mathbb{R})$, we have the map u_a of Proposition 4, which satisfies to axioms i) and ii) of Definition 1. Also it is not difficult to check that axiom iii) is satisfied, with $\alpha(u_a) \le ||a||$. The trouble arises with axiom iv): indeed we have $\beta(u_a) = \sup_{x\in G} ||ax|| = +\infty$. The next step should be to use other metrics on G, but recall that $GL_n(\mathbb{R})$ admits no two-sided invariant metric compatible with its topology (see e.g. [4]). However we can consider the right invariant metric defined by $d(x, y) = \log(1 + ||xy^{-1} - 1|| + ||yx^{-1} - 1)||)$ (see [4], 8.15). Then u_a satisfies to axioms i), ii) and iv), but we have not been able to check that axiom iii) holds, and we suspect that it does not. Finally consider the left invariant metric defined by $d(x, y) = \log(1 + ||x^{-1}y - 1|| + ||y^{-1}x - 1)||)$. Then u_a satisfies to axioms i), ii) and iii), but the fourth axiom does not hold.

4. Mutational structure on homogeneous spaces

Our next goal is to construct a mutational structure on the unit sphere of an Euclidean space. Such a sphere is an homogeneous space over the orthogonal group, so this suggests to study the more general situation of a space on which a compact Lie group acts. Once again we do this in the framework of quasi-exponential maps.

The following result is in fact a generalization of a part of Proposition 4 and of Corollary 5, the proof being exactly the same.

Proposition 7. *Let G be a metric group with a left and right invariant metric d_G and let $q : V \longrightarrow G$ be a quasi-exponential map. Let (X, d_X) be a metric space. Assume that G acts continuously and isometrically on X (i.e. $d_X(gx, gy) = d_X(x, y)$ for $g \in G$ and $x, y \in X$), and that $d_X(gx, hx) \leq d_G(g, h)$, for $g, h \in G$ and $x \in X$. For $a \in V$, define a map*

$$\begin{aligned} u_a : X \times [0,1] &\longrightarrow X \\ (x,h) &\longmapsto q(ha)x \end{aligned}$$

Then u_a is a transition over X, with $\alpha(u_a) = 0$ and $\beta(u_a) \leq ||a||$. Put $\mathcal{D}_{X,q} = \{u_a,\ a \in V\}$. If furthermore we have

$$\sup_{x \in X} \limsup_{h \to 0^+} \frac{d_X(q(ha)x, q(hb)x)}{h} = ||a - b||, \quad \forall a, b \in V,\ \forall x \in X,$$

then $(X, \mathcal{D}_{X,q})$ is a mutational space, and the metric spaces $\mathcal{D}_{X,q}$ and V are isometric. $\square$

For $n \in \mathbb{N}^*$, consider now the unit sphere S^{n-1} of the the Euclidean space $\mathbb{R}^n$. Then the orthogonal group $O(n)$ acts on S^{n-1}, and the conditions of Proposition 7 are easily seen to be fulfilled. Thus we get a mutational space structure on S^{n-1}, with transition space isometric to $\mathfrak{so}(n)$:

Proposition 8. *Let $a \in \mathfrak{so}(n)$. Then the map*

$$\begin{aligned} u_a : S^{n-1} \times [0,1] &\longrightarrow S^{n-1} \\ (x,t) &\longmapsto \exp(ta)(x) \end{aligned}$$

is a transition over S^{n-1}, and $(S^{n-1}, \mathcal{D}_{S^{n-1}})$ is a mutational space, with $\mathcal{D}_{S^{n-1}} = \{u_a,\ a \in \mathfrak{so}(n)\}$. $\square$

Just as in the preceding section, we conclude with a disappointment. Endow $\mathbb{R}^n$ with the norm $||.||_p$ for $p \in [1, \infty]$. The isometry group of the unit sphere S_p^{n-1} is finite if $p \neq 2$, so its Lie algebra is trivial, and hence in this case the mutational structure provided by Proposition 7 is trivial.

5. Mutational structure on orbit spaces

We now wish to construct a mutational structure on the projective space $\mathbb{P}^n(\mathbb{R})$. Since $\mathbb{P}^n(\mathbb{R})$ is the orbit space $S^n/\mathbb{Z}_2$ of the Euclidean sphere S^n for the antipodal action $x \longmapsto -x$ of the cyclic group of order 2, we consider the following question: given a mutational space $(X, \mathcal{D}_X)$ and an action of a group G on X, is it possible to endow the orbit space X/G with a natural mutational space structure ? The positive answer to this question will furnish many new examples.

Let us consider an isometric action of a group G on a metric space X. The orbit space is denoted by X/G, and the class of an element $x \in X$ is denoted by $\bar{x}$ in X/G, with $\bar{x} = \bar{y} \iff \exists g \in G$ such that $y = gx$. We have a well-defined map

$$\begin{aligned} \bar{d} : X/G \times X/G &\longrightarrow \mathbb{R} \\ (\bar{x}, \bar{y}) &\longmapsto \inf_{g \in G} d(gx, y) \end{aligned}$$

which is easily seen to be a semi-distance (see e.g. [4], 8.14, this is very classical). If in addition each G-orbit is closed in X, then $\bar{d}$ is a distance on X/G, and then the quotient topology on X/G is the one induced by the metric $\bar{d}$. Let us now propose the following definition for the concept of a group action on a mutational space.

Definition 9. *Let G be a topological group and let $(X, \mathcal{D}_X)$ be a mutational space. A **mutational action** of G on $(X, \mathcal{D}_X)$ consists of a continuous and isometric action of G on X satisfying the following conditions.*
i) For $x \in X$ and $\varepsilon > 0$, the subset $G_{x,\varepsilon} = \{g \in G \mid \bar{B}_\varepsilon(gx) \cap \bar{B}_\varepsilon(x) \neq \emptyset\}$ is relatively compact in G.
ii) For $u \in \mathcal{D}_X$, $g \in G$ and $t \in [0, 1]$, we have $u(gx, t) = gu(x, t)$.

Note that condition i) is closely related to the concept of proper action. When G is compact, this condition is always satisfied, while when G is discrete, it is satisfied if and only if each $G_{x,\varepsilon}$ is finite. We did not assume that each orbit is closed, so we still have to check that $\bar{d}$ is a distance. This follows immediately from the following lemma, and the quotient topology on X/G will be the one induced by the metric $\bar{d}$.

Lemma 10. *Let G be a topological group acting continuously and isometrically on a metric space X. Assume that condition i) of Definition 9 is satisfied. Then for all $x, y \in X$, there exists $g \in G$ such that $\bar{d}(\bar{x}, \bar{y}) = d(gx, y)$.*

Proof. We can assume that $x \neq y$. Put $\varepsilon = 2d(x, y)$. Let $g \notin G_{x,\varepsilon}$, then $\bar{B}_\varepsilon(gx) \cap \bar{B}_\varepsilon(x) = \emptyset$, so $d(gx, x) > \varepsilon$. Then we have

$$2d(x, y) = \varepsilon < d(gx, x) \leq d(gx, y) + d(y, x) \Rightarrow d(gx, x) > d(x, y) \geq \bar{d}(\bar{x}, \bar{y}).$$

Hence $\bar{d}(\bar{x}, \bar{y}) = \inf_{g \in G_{x,\varepsilon}} d(gx, y)$ and

$$\inf_{g \in G} d(gx, y) \leq \inf_{g \in \overline{G_{x,\varepsilon}}} d(gx, y) \leq \inf_{g \in G_{x,\varepsilon}} d(gx, y) = \inf_{g \in G} d(gx, y) = \bar{d}(\bar{x}, \bar{y}).$$

Thus we have our result, since $\overline{G_{x,\varepsilon}}$ is compact. □

Now we can construct transitions on the orbit space of a mutational space.

Proposition 11. *Let G be topological group. Assume that we have a mutational action of G on a mutational space $(X, \mathcal{D}_X)$. Then any $u \in \mathcal{D}_X$ induces a continuous map*

$$\begin{aligned} \bar{u} : X/G \times [0, 1] &\longrightarrow X/G \\ (\bar{x}, h) &\longmapsto \overline{u(x, h)}, \end{aligned}$$

and $\bar{u}$ is a transition on X/G with $\alpha(\bar{u}) \leq \alpha(u)$ and $\beta(\bar{u}) \leq \beta(u)$. For $u, v \in \mathcal{D}_X$, we have $d_\Delta(\bar{u}, \bar{v}) \leq d_\Delta(u, v)$.

Proof. It is clear, thanks to condition ii) in Definition 9, that $\bar{u}$ is well defined, and is continuous. It is obvious that axiom i) in Definition 1 is satisfied, and it is easy to see, using $\bar{d}(\bar{x}, \bar{y}) \leq d(x, y)$, $\forall x, y \in X$, that axioms ii) and iv) are also satisfied, with $\beta(\bar{u}) \leq \beta(u)$. Let $x, y \in X$ with $\bar{x} \neq \bar{y}$. By Lemma 10, there exists $g \in G$

such that $\bar{d}(\bar{x},\bar{y}) = d(gx,y)$, and since $\bar{d}(\bar{u}(\bar{x},h),\bar{u}(\bar{y},h)) \leq d(gu(x,h),u(y,h)) = d(u(gx,h),u(y,h))$, we easily see that condition iii) in Definition 1 is satisfied, with $\alpha(\bar{u}) \leq \alpha(u)$. Hence $\bar{u}$ is a transition, and the last assertion is shown in the same way. □

Thus, given a mutational action of a topological group on a mutational space $(X, \mathcal{D}_X)$, we have a subset $\mathcal{D}_{X/G} := \{\bar{u}, u \in \mathcal{D}_X\} \subset \mathcal{U}(X/G)$. The following question arises now: is $\mathcal{D}_{X/G}$ closed in $\mathcal{U}(X/G)$? Since the map $\mathcal{D}_X \longrightarrow \mathcal{D}_{X/G}$, $u \longmapsto \bar{u}$, is continuous, we have the following result:

Proposition 12. *Let G be topological group, and assume that we have a mutational action of G on a mutational space $(X, \mathcal{D}_X)$ with a compact transition space $\mathcal{D}_X$. Then $(X/G, \mathcal{D}_{X/G})$ is a mutational space.* □

The condition that $\mathcal{D}_X$ is compact arises in some examples, but we would like to have a milder assumption, e.g. that $\mathcal{D}_X$ is complete. For this, we will make the additional assumption that the group G is discrete. We need two lemmas. The first one is classical in the context of proper actions.

Lemma 13. *Let G be a discrete group acting continuously and isometrically on a metric space X. Assume that condition i) of Definition 9 is satisfied. Then the G-orbits are discrete: $\forall x \in X$, $\exists \varepsilon > 0$ such that $Gx \cap B_\varepsilon(x) = \{x\}$.*

Proof. Let us first prove that $\forall x, y \in X$, $\forall \varepsilon > 0$, the set $Gy \cap \bar{B}_\varepsilon(x)$ is finite. Let $\alpha = \max(\varepsilon, d(x,y))$. Let $g \in G$ be such that $gy \in \bar{B}_\varepsilon(x)$. Then $y \in \bar{B}_\varepsilon(g^{-1}x) \Rightarrow y \in \bar{B}_\alpha(g^{-1}x)$, and since $d(x,y) \leq \alpha$ we also have $y \in \bar{B}_\alpha(x)$. Hence $g^{-1} \in G_{x,\alpha}$, which is finite by condition i) of Definition 9, and we have proved our assertion.

Now let $x \in X$ and $t > 0$. Then there exists only a finite number of elements $g \notin \mathrm{Stab}(x)$ such that $gx \in B_t(x)$ (recall that $\mathrm{Stab}(x) = \{g \in G \mid gx = x\}$). Let $g_1, \ldots, g_n$ be those elements. Choosing ε such that $0 < \varepsilon < d(g_i, x)$, $\forall i$, we have $B_\varepsilon(x) \cap Gx = \{x\}$. □

Lemma 14. *Let G be a discrete group acting continuously and isometrically on a metric space X. Assume that condition i) of Definition 9 is satisfied. Let $x \in X$. Then there exists $\varepsilon > 0$ such that $\forall y, z \in B_\varepsilon(x)$, $\forall g \notin \mathrm{Stab}(x)$, we have*

$$d(gy,z) \geq d(y,z) \geq \bar{d}(\bar{y},\bar{z}).$$

In particular if $y, z \in B_\varepsilon(x)$ and $\mathrm{Stab}(x) \subset \mathrm{Stab}(y)$, then $\bar{d}(\bar{y},\bar{z}) = d(y,z)$.

Proof. Let $\alpha > 0$ be such that $Gx \cap B_\alpha(x) = \{x\}$ and $Gx \cap B_\alpha(y) = \{y\}$ (Lemma 13). Let $\varepsilon = \frac{\alpha}{4}$. Let $g \notin \mathrm{Stab}(x)$: then $d(gx,x) \geq \alpha$. Let $y, z \in B_\varepsilon(x)$. We have

$$\alpha \leq d(gx,x) \leq d(gx,gy) + d(gy,z) + d(z,x) < \frac{\alpha}{4} + d(gy,z) + \frac{\alpha}{4} = \frac{\alpha}{2} + d(gy,z).$$

Thus $d(gy,z) \geq \frac{\alpha}{2} \geq d(y,z)$, and our result follows. □

It is now easy to prove the following result.

Proposition 15. *Let G be discrete group, and assume that we have a mutational action of G on a mutational space $(X, \mathcal{D}_X)$ with a complete transition space $\mathcal{D}_X$. Then $(X/G, \mathcal{D}_{X/G})$ is a mutational space, and the metric spaces $\mathcal{D}_X$ and $\mathcal{D}_{X/G}$ are isometric.*

Proof. Let $u, v \in \mathcal{D}_X$. We just have to check that $d_\Delta(u,v) = d_\Delta(\bar{u},\bar{v})$. Let $x \in X$, and choose $\varepsilon > 0$ as in Lemma 14. Then for h small enough, we have $u(x,h), v(x,h) \in B_\varepsilon(x)$ and since $\mathrm{Stab}(x) \subset \mathrm{Stab}(u(x,h))$, we have $\bar{d}(\overline{u(x,h)}, \overline{v(x,h)}) = d(u(x,h), v(x,h))$ by Lemma 14. Hence it is clear that $d_\Delta(u,v) = d_\Delta(\bar{u},\bar{v})$. □

In this way this last proposition and Proposition 8 furnish a mutational structure on the projective space $\mathbb{P}^n(\mathbb{R}) = S^n/\mathbb{Z}_2$. We can construct many other examples, e.g. embedding any finite group of order n in an orthogonal group $O(n)$: we get a mutational structure on the quotient spaces $\mathbb{R}^n/G$ or S^{n-1}/G.

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