

HOPF ALGEBRA DEFORMATIONS OF BINARY POLYHEDRAL GROUPS

JULIEN BICHON AND SONIA NATALE

ABSTRACT. We show that semisimple Hopf algebras having a self-dual faithful irreducible comodule of dimension 2 are always obtained as abelian extensions with quotient \mathbb{Z}_2 . We prove that nontrivial Hopf algebras arising in this way can be regarded as deformations of binary polyhedral groups and describe its category of representations. We also prove a strengthening of a result of Nichols and Richmond on cosemisimple Hopf algebras with a 2-dimensional irreducible comodule in the finite dimensional context. Finally, we give some applications to the classification of certain classes of semisimple Hopf algebras.

1. INTRODUCTION AND MAIN RESULTS

Throughout the paper we work over k , an algebraically closed field of characteristic zero. This paper is a contribution to the study of low dimensional semisimple Hopf algebras, namely those that admit a faithful comodule of dimension 2. Our starting point is a theorem by Nichols and Richmond [38], that we recall first.

Let H be a cosemisimple Hopf algebra over k and let $C \subseteq H$ be a simple subcoalgebra of dimension 4 (this is equivalent to say that H has an irreducible comodule of dimension 2). The main result of the paper [38] says that, in this case, at least one of the following possibilities holds:

- (1) H has a group-like element g of order 2 such that $gC = C$;
- (2) H has a Hopf subalgebra of dimension 24, which contains a group-like element g of order 2 such that $gC \neq C$;
- (3) H has a Hopf subalgebra of dimension 12 or 60;
- (4) H has a family $\{C_n : n \geq 1\}$ of simple subcoalgebras such that $\dim C_n = n^2$, and for all $n \geq 2$,

$$C_n \mathcal{S}(C_2) = C_{n-1} + C_{n+1}, \quad n \text{ even}, \quad C_n C_2 = C_{n-1} + C_{n+1}, \quad n \text{ odd}.$$

See [38, Theorem 11]. Note that if H is finite dimensional only the possibilities (1)–(3) can arise, and then $\dim H$ is even (in particular this gives a partial positive answer to one of Kaplansky's long-standing conjectures).

Date: May 21, 2010.

1991 *Mathematics Subject Classification.* 16W30.

This work was partially supported by CONICET–CNRS, ANPCyT, SeCYT (UNC), FaMAF, Alexander von Humboldt Foundation, and the ANR project Galoisint.

In this paper we obtain a more precise description of this result, in case H is finite dimensional. We prove in Subsection 3.3 the following theorem that strengthens the Nichols-Richmond Theorem above in the finite dimensional context, and connects this class of Hopf algebras with polyhedral groups.

Theorem 1.1. *Let H be a cosemisimple finite dimensional Hopf algebra. Suppose H contains a simple subcoalgebra C of dimension 4. Then the subalgebra $B = k[\mathcal{CS}(C)]$ is a commutative Hopf subalgebra of H isomorphic to k^Γ , where Γ is a non cyclic finite subgroup of $\mathrm{PSL}_2(k)$ of even order.*

Furthermore, let $\chi \in C$ be the irreducible character contained in C , and let $G[\chi] \subseteq G(H)$ be the stabilizer of χ with respect to left multiplication. Then $|G[\chi]|$ divides 4, and the following hold:

- (i) If $|G[\chi]| = 4$, then $B \simeq k^{\mathbb{Z}_2 \times \mathbb{Z}_2}$.
- (ii) If $|G[\chi]| = 2$, then $B \simeq k^{D_n}$, where $n \geq 3$.
- (iii) If $|G[\chi]| = 1$, then $B \simeq k^{A_4}$, k^{S_4} , or k^{A_5} .

Observe that, in addition, the category of B -comodules is contained in the adjoint subcategory of the fusion category H – comod of finite dimensional H -comodules.

In special situations, the above theorem may be refined as follows. Indeed, assume that the irreducible 2-dimensional comodule V corresponding to C satisfies $V^* \otimes V \simeq V \otimes V^*$. Then the Hopf subalgebra H' of H generated by C fits into a cocentral exact sequence

$$k \rightarrow k^\Gamma \rightarrow H' \rightarrow k\mathbb{Z}_m \rightarrow k$$

for a polyhedral group Γ of even order and $m \geq 1$. See Theorem 3.5.

Moreover when V is self-dual, the Hopf algebra $H' = k[C]$ generated by C can be completely described. See Theorem 1.2 below.

Compact quantum subgroups of $\mathrm{SU}_{-1}(2)$, or in other words, quotient Hopf $*$ -algebras $\mathcal{O}_{-1}[\mathrm{SU}_2(\mathbb{C})] \rightarrow H$, were determined in [43]. In this paper we give an algebraic version of this result. We obtain a description, using Hopf algebra exact sequences, of quotient Hopf algebras $\mathcal{O}_{-1}[\mathrm{SL}_2(k)] \rightarrow H$ in the finite dimensional case. See Theorem 5.19. Our approach gives a more precise description of such quotients, and also implies a description of their tensor categories of representations in terms of \mathbb{Z}_2 -equivariantizations of certain pointed fusion categories. See Corollary 5.12.

It turns out that every such Hopf algebra H can be regarded as a deformation of an appropriate binary polyhedral group. We remark that such groups admit no nontrivial cocycle deformation in the sense of [33, 10], since every Sylow subgroup is either generalized quaternion or cyclic.

We combine this result with the classification of Hopf algebras associated to non-degenerate bilinear forms in the 2×2 case [7, 12], to prove the following theorem.

Theorem 1.2. *Let H be a semisimple Hopf algebra over k . Assume H has a self-dual faithful irreducible comodule V of dimension 2. Let $\nu(V) = \pm 1$ denote the Frobenius-Schur indicator of V .*

Then we have:

- (i) *Suppose $\nu(V) = -1$. Then H is commutative. Moreover, $H \simeq k^{\tilde{\Gamma}}$, where $\tilde{\Gamma}$ is a non-abelian binary polyhedral group.*
- (ii) *Suppose $\nu(V) = 1$. Then either H is commutative and isomorphic to k^{D_n} , $n \geq 3$, or H is isomorphic to one of the nontrivial Hopf algebra deformations $\mathcal{A}[\tilde{\Gamma}]$ or $\mathcal{B}[\tilde{\Gamma}]$ of the binary polyhedral group $\tilde{\Gamma}$, as in Subsection 5.4.*

Let $C \subseteq H$ be the simple subcoalgebra containing V . Note that the assumptions on V in Theorem 1.2 amount to the assumption that $\dim C = 4$, $\mathcal{S}(C) = C$ and C generates H as an algebra.

Theorem 1.2 is proved in Subsection 5.4. We point out that only one nontrivial deformation, $\mathcal{B}[\tilde{\Gamma}]$, can occur in the case of the binary icosahedral group $\tilde{\Gamma}$. The algebra and coalgebra structures of the Hopf algebras $\mathcal{A}[\tilde{\Gamma}]$, $\mathcal{B}[\tilde{\Gamma}]$, are described in Remark 5.22. Their fusion rules are the same as the ones of $\tilde{\Gamma}$, see Remark 5.23.

In particular, if H is one of the Hopf algebras in (ii), then $\dim H = |\tilde{\Gamma}|$ can be either 24, 48, 120 or $4n$, $n \geq 2$. It turns out that, when $k = \mathbb{C}$ is the field of complex numbers, such Hopf algebras admit a Kac algebra structure.

Furthermore, let Vec^Γ be the fusion category of Γ -graded vector spaces, where $\Gamma \simeq \tilde{\Gamma}/Z(\tilde{\Gamma})$ is the corresponding polyhedral group. Then we have an equivalence of fusion categories $\text{Rep } H \simeq (\text{Vec}^\Gamma)^{\mathbb{Z}_2}$, where the last category is the equivariantization of the category Vec^Γ with respect to an appropriate action of \mathbb{Z}_2 .

The Hopf algebras $\mathcal{A}[\tilde{D}_n]$, $\mathcal{B}[\tilde{D}_n]$, $n \geq 3$, were constructed and studied by Masuoka in [31]. By the results in *loc. cit.*, $\mathcal{A}[\tilde{D}_n]$ is the only nontrivial cocycle deformation of the commutative Hopf algebra $k^{D_{2n}}$. On the other hand, the Hopf algebra $\mathcal{B}[\tilde{D}_n]$ is self-dual and it has no nontrivial cocycle deformations.

In the case when $n = 2$, $\mathcal{A}[\tilde{D}_2] \simeq \mathcal{B}[\tilde{D}_2]$, and they are both isomorphic to the 8-dimensional Kac-Paljutkin Hopf algebra H_8 .

We give some applications of the main results to the classification of a special class of semisimple Hopf algebras. We consider a semisimple Hopf algebra H such that $\deg \chi \leq 2$, for all irreducible character $\chi \in H$.

We show in Theorem 6.4 that such a Hopf algebra H is not simple. Moreover, we prove the existence of certain exact sequences for H , which imply that either H or H^* contains a nontrivial central group-like element and, in addition, that H is lower semisolvable. See Corollary 6.6.

An application to semisimple Hopf algebras of dimension 60 is also given in Subsection 6.2. We show that for two known examples of simple Hopf

algebras of this dimension, the Hopf algebras are determined by its coalgebra types.

The paper is organized as follows: in Section 2 we recall some facts on Hopf algebra extensions and their relations with the categories of representations. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we recall the definition and basic properties of the Hopf algebra of a non-degenerate bilinear form and its relation with the classification problem we consider. We study in this section a canonical \mathbb{Z}_2 -grading in the category of finite dimensional comodules. In Section 5 we discuss the 2×2 case and further facts related to the Hopf algebra $\mathcal{O}_{-1}[\mathrm{SL}_2(k)]$ and the classification of its finite dimensional quotients. Section 6 contains some applications to the classification of certain classes of semisimple Hopf algebras. We include at the end an Appendix with some facts concerning a special kind of Opext groups that include those related to the Hopf algebras $\mathcal{A}[\tilde{\Gamma}]$, $\mathcal{B}[\tilde{\Gamma}]$.

Acknowledgement. The authors thank Florian Steiner for pointing out a mistake in a previous version of the proof of Proposition 3.2. The research of S. Natale was done during a stay at the Mathematisches Institut der Universität München, as an Alexander von Humboldt Fellow. She thanks Prof. Hans-Jürgen Schneider for the kind hospitality.

2. PRELIMINARIES

2.1. Hopf algebra exact sequences. Recall that a sequence of Hopf algebra maps

$$(2.1) \quad k \rightarrow K \xrightarrow{i} H \xrightarrow{p} \overline{H} \rightarrow k,$$

is called *exact* if the following conditions hold:

- (1) i is injective and p is surjective,
- (2) $p \circ i = \epsilon 1$,
- (3) $\ker p = HK^+$,
- (4) $K = H^{\mathrm{cop}} = \{h \in H : (\mathrm{id} \otimes p)\Delta(h) = h \otimes 1\}$.

Assume we have a sequence (2.1). If H is faithfully flat over K (in particular if H is finite dimensional), then (1), (2) and (3) imply (4). On the other hand, if H is faithfully coflat over \overline{H} , then (1), (2) and (4) imply (3). See e.g. [1].

We shall say that the Hopf algebra map $p : H \rightarrow \overline{H}$ is *cocentral* if $p(h_1) \otimes h_2 = p(h_2) \otimes h_1$, for all $h \in H$.

Lemma 2.1. *Let H be a Hopf algebra. Let also G be a finite group and let $p : H \rightarrow kG$ be a surjective Hopf algebra map. Suppose p is cocentral. Then there is an exact sequence of Hopf algebras $k \rightarrow K \rightarrow H \xrightarrow{p} kG \rightarrow k$, where $K = H^{\mathrm{cop}}$.*

Proof. Let $I = \ker p$. Then I is a coideal of H of finite codimension. Since p is cocentral by assumption, then I is left and right conormal. Also, $K = H^{\text{cop}} = {}^{\text{cop}}H$ is a Hopf subalgebra of H , and thus we have a sequence of Hopf algebra maps (2.1), that satisfies (1), (2) and (4). By [45, 2.1], H is left and right faithfully coflat over $H/I \simeq kG$. Hence the sequence is exact, as claimed. \square

A Hopf subalgebra K of H is called *normal* if it is stable under both left and right adjoint actions of H on itself, defined, respectively, by

$$\text{ad}_l(h)(x) = h_1 x \mathcal{S}(h_2), \quad \text{ad}_r(h)(x) = \mathcal{S}(h_1) x h_2,$$

for all $h, x \in H$. If $K \subseteq H$ is a normal Hopf subalgebra, then the ideal HK^+ is a Hopf ideal and the canonical map $H \rightarrow \overline{H} := H/HK^+$ is a Hopf algebra map. Hence, if H is faithfully flat over K , there is an exact sequence (2.1) where all maps are canonical. A Hopf algebra is said to be simple if it has no proper normal Hopf subalgebra.

Suppose, from now on, that H is finite dimensional. Then the exact sequence above is cleft and H is isomorphic to a bicrossed product $H \simeq K \#_{\sigma}^{\tau} \overline{H}$, for appropriate compatible data.

The exact sequence (2.1) is called *abelian* if K is commutative and \overline{H} is cocommutative. In this case $K \simeq k^N$ and $\overline{H} \simeq kF$, for some finite groups N and F ; in particular, H is semisimple.

As explained in [29, Section 4] such an extension is determined by a matched pair (F, N) with respect to compatible actions $\triangleright: N \times F \rightarrow F$ and $\triangleleft: N \times F \rightarrow N$, and 2-cocycles $\sigma: kF \otimes kF \rightarrow (k^N)^{\times}$ and $\tau: kN \otimes kN \rightarrow (k^F)^{\times}$, subject to appropriate compatibility conditions. For a fixed matched pair (F, N) , the isomorphism classes of extensions $k \rightarrow k^N \rightarrow H \rightarrow kF \rightarrow k$, or equivalently, the equivalence classes of pairs (σ, τ) , form an abelian group that we shall denote $\text{Opext}(k^N, kF)$.

Since they will appear later on in this paper, we recall here the explicit formulas for the multiplication, comultiplication and antipode in the bicrossed product $k^{N\tau} \#_{\sigma} kF$ representing the class of (σ, τ) in $\text{Opext}(k^N, kF)$:

$$(2.2) \quad (e_s \# x)(e_t \# y) = \delta_{s \triangleleft x, t} \sigma_s(x, y) e_s \# xy,$$

$$(2.3) \quad \Delta(e_s \# x) = \sum_{gh=s} \tau_x(g, h) e_g \# (h \triangleright x) \otimes e_h \# x,$$

$$(2.4) \quad \mathcal{S}(e_g \# x) = \sigma_{(g \triangleleft x)^{-1}}((g \triangleright x)^{-1}, g \triangleright x)^{-1} \tau_x(g^{-1}, g)^{-1} e_{(g \triangleleft x)^{-1}} \# (g \triangleright x)^{-1},$$

for all $s, t \in N$, $x, y \in F$, where $\sigma_s(x, y) = \sigma(x, y)(s)$ and $\tau_x(s, t) = \tau(s, t)(x)$.

Remark 2.2. Suppose that the action $\triangleright: N \times F \rightarrow F$ is trivial. Then it follows from formula (2.3) that the subspaces $R_x := k^N \# x$ are subcoalgebras of $H = k^{N\tau} \#_{\sigma} kF$ and we have a decomposition of coalgebras $H = \bigoplus_{x \in F} R_x$.

Moreover, in this case, $\tau_x : N \times N \rightarrow k^\times$ is a 2-cocycle on N and, as coalgebras, $k^N \# x \simeq (k_{\tau_x} N)^*$, where $k_{\tau_x} N$ denotes the twisted group algebra.

2.2. Type of a cosemisimple Hopf algebra. For a Hopf algebra H we shall denote by $H\text{-comod}$ the tensor category of its finite dimensional (right) comodules.

Let H be a finite dimensional cosemisimple Hopf algebra over k . As a coalgebra, H is isomorphic to a direct sum of full matrix coalgebras

$$(2.5) \quad H \simeq k^{(n)} \oplus \bigoplus_{d_i > 1} M_{d_i}(k)^{(n_i)},$$

where $n = |G(H)|$.

If we have an isomorphism as in (2.5), we shall say that H is of *type* $(1, n; d_1, n_1; \dots; d_r, n_r)$ as a coalgebra. If H^* , which is also cosemisimple, is of type $(1, n; d_1, n_1; \dots)$ as a coalgebra, we shall say that H is of *type* $(1, n; d_1, n_1; \dots)$ as an algebra.

So that H is of type $(1, n; d_1, n_1; \dots; d_r, n_r)$ as a (co-)algebra if and only if H has n non-isomorphic one-dimensional (co-)representations, n_1 non-isomorphic irreducible (co-)representations of dimension d_1 , etc.

By an *irreducible character* $\chi \in H$, we shall mean the character $\chi = \chi_V$ of an irreducible corepresentation V of H . Hence, χ_V is a cocommutative element of H , such that $f(\chi) = \text{Tr}_V(f)$, for all $f \in H^*$, regarding V as an irreducible representation of H^* . The dimension of V will be called the *degree* of χ , denoted $\deg \chi$.

Every irreducible character $\chi \in H$ is contained in a unique simple subcoalgebra. If this subcoalgebra is isomorphic to $M_{d_i}(k)$, then $d_i = \deg \chi$.

Suppose H is of type $(1, n; d_1, n_1; \dots; d_r, n_r)$ as a coalgebra. It follows from the Nichols-Zoeller Theorem [39], that n divides both $\dim H$ and $n_i d_i^2$, for all i . Moreover, if $d_i = 2$ for some i , then the dimension of H is even [38]. Also, by [54], if $n = 1$, then $\{d_i : d_i > 1\}$ has at least three elements.

2.3. The adjoint subcategory. Let \mathcal{C} be a k -linear abelian semisimple tensor category over k . Recall from [14, 8.5.] that the *adjoint subcategory* \mathcal{C}_{ad} is the full subcategory generated by the simple components of $V \otimes V^*$, where V runs over the set of (isomorphism classes of) simple objects of \mathcal{C} .

Suppose $\mathcal{C} = \text{Rep } H$ is the fusion category of finite dimensional representations of H , where H is a semisimple (hence finite dimensional) Hopf algebra over k .

In this case there exists a unique maximal *central* Hopf subalgebra K of H , such that $\mathcal{C}_{\text{ad}} = \text{Rep } H_{\text{ad}}$, where $H_{\text{ad}} = H/HK^+$ is the corresponding quotient Hopf algebra. Moreover, one has $K = k^{U(\mathcal{C})}$, where $U(\mathcal{C})$ is the *universal grading group* of \mathcal{C} . See [18, Theorem 3.8]. In particular, there is a central extension of Hopf algebras

$$k \rightarrow k^{U(\mathcal{C})} \rightarrow H \rightarrow H_{\text{ad}} \rightarrow k.$$

There is also a dual construction associated to the category $H - \text{comod}$, for H cosemisimple. Indeed, the Hopf subalgebra H_{coad} of H determined by $H_{\text{coad}} - \text{comod} = (H - \text{comod})_{\text{ad}}$ gives a universal cocentral exact sequence

$$(2.6) \quad k \rightarrow H_{\text{coad}} \rightarrow H \rightarrow kU(\mathcal{C}') \rightarrow k$$

where $\mathcal{C}' = H - \text{comod}$.

Note that the notion of simplicity of a finite dimensional Hopf algebra is self-dual. As a consequence of the above, we get the following lemma:

Lemma 2.3. *Suppose H is simple and not commutative. Then $H_{\text{ad}} = H = H_{\text{coad}}$. \square*

In other words, if H is simple, then H is generated as algebra by the simple subcoalgebras appearing in the decomposition of $\mathcal{CS}(C)$, where C runs over the simple subcoalgebras of H .

Lemma 2.4. *Suppose H contains a faithful irreducible character χ . Assume in addition that $\chi\chi^* = \chi^*\chi$. Then $H_{\text{coad}} = k[\mathcal{CS}(C)]$, where $C \subseteq H$ is the simple subcoalgebra containing χ .*

Proof. We have $k[\mathcal{CS}(C)] \subseteq H_{\text{coad}}$. The faithfulness assumption on χ means that $H = k[C]$ and for every irreducible character λ of H , using the assumption $\chi\chi^* = \chi^*\chi$, there exist nonnegative integers m and p such that λ appears with positive multiplicity in $\chi^m(\chi^*)^p$. Hence λ^* appears with positive multiplicity in $\chi^p(\chi^*)^m$ (since H is cosemisimple, we have $\chi^{**} = \chi$). Thus $\lambda\lambda^*$ appears with positive multiplicity in $\chi^{m+p}(\chi^*)^{m+p}$. Then $\lambda\lambda^* \in k[\mathcal{CS}(C)]$. Since this holds for any irreducible character λ , we get $H_{\text{coad}} \subseteq k[\mathcal{CS}(C)]$ and the lemma follows. \square

3. ON A THEOREM OF NICHOLS AND RICHMOND

In this section we aim to give a proof of our most general result on finite dimensional cosemisimple Hopf algebras with a simple comodule of dimension 2, strengthening, in this context, the Nichols-Richmond Theorem [38]. The proof is based on the algebraic generalization of some well known results in the theory of compact quantum groups [2, 3].

3.1. Comodule algebras of dimension 4. Let V be a finite dimensional H -comodule. Then $\text{End } V \simeq V \otimes V^*$ is an H -comodule algebra with the diagonal coaction of H and multiplication given by composition.

Remark 3.1. Under the canonical (H -colinear) isomorphism $V \otimes V^* \simeq \text{End } V$, the trace map $\text{Tr} : \text{End } V \rightarrow k$ becomes identified with the evaluation map $V \otimes V^* \rightarrow k$, $v \otimes f \rightarrow f(v)$. If H is involutory, that is, if $\mathcal{S}^2 = \text{id}$, then $\text{Tr} : \text{End } V \rightarrow k$ is an H -comodule map.

Suppose H is cosemisimple and $\mathcal{S}^2 = \text{id}$. Let $V \not\cong k$ be a simple (hence finite dimensional) H -comodule, and decompose $V \otimes V^* = k \oplus W$, where $W \subseteq V$ is an H -subcomodule. Then we have $W = \ker \text{Tr}$.

Indeed, the restriction $\text{Tr}|_W$ is an H -comodule map. Therefore $\text{Tr}|_W = 0$, since $\text{Hom}^H(W, k) = 0$, $V \not\cong k$ being simple.

The following result is related to the computation of the compact quantum symmetry group of the 2×2 complex matrix algebra given in [3].

Proposition 3.2. *Let H be a Hopf algebra and let A be an H -comodule algebra such that $A \simeq M_2(k)$. Assume that the trace map $\text{Tr} : A \rightarrow k$ is an H -comodule map. Then the subalgebra $k[C_A]$ generated by the subcoalgebra C_A of matrix coefficients of A is a commutative Hopf subalgebra of H .*

We have moreover $k[C_A] \simeq \mathcal{O}[\Gamma]$, where Γ is an algebraic subgroup of $\text{PSL}_2(k)$.

Proof. Let $\rho : A \rightarrow A \otimes H$ be the H -comodule structure on A . Consider the basis of A consisting of the unit quaternions

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_1 = \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

These satisfy the following multiplication rules:

$$(3.1) \quad e_i^2 = -e_0, \quad 1 \leq i \leq 3, \quad e_1 e_2 = e_3, \quad e_2 e_3 = e_1, \quad e_3 e_1 = e_2.$$

Moreover, $\{e_1, e_2, e_3\}$ forms a basis of W , the kernel of the trace map $\text{Tr} : M_2(k) \rightarrow k$. Since by assumption, this is an H -subcomodule of A , then we can write $\rho(e_0) = e_0 \otimes 1$ and

$$\rho(e_i) = \sum_{j=1}^3 e_j \otimes x_{ji},$$

for all $1 \leq i \leq 3$, where $x_{ij} \in H$. In particular, x_{ij} , $1 \leq i, j \leq 3$, generate the subalgebra $k[C_A]$. We have moreover,

$$(3.2) \quad \Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}, \quad 1 \leq i, j \leq 3.$$

The map $W \otimes W \rightarrow k$, $e_i \otimes e_j \mapsto \delta_{ij}$, is H -colinear since it is the composition of H -colinear maps

$$W \otimes W \subset M_2(k) \otimes M_2(k) \xrightarrow{\text{mult}} M_2(k) \xrightarrow{-\frac{1}{2}\text{Tr}} k.$$

Hence we see that the matrix (x_{ij}) is orthogonal, and we have $S(x_{ij}) = x_{ji}$, $1 \leq i, j \leq 3$ (see Section 4 for details).

Since ρ is an algebra map, it must preserve the relations (3.1). The relations $\rho(e_i)^2 = -e_0 \otimes 1$, $1 \leq i \leq 3$, amount to

$$x_{li} x_{ti} = x_{ti} x_{li}$$

for all $i = 1, 2, 3$. This means that the matrix coefficients in a given column commute pairwise. Using the antipode, we get

$$x_{li} x_{lj} = x_{lj} x_{li}$$

and thus the matrix coefficients in a given row commute pairwise.

The remaining relations (3.1) amount to

$$(3.3) \quad \varepsilon_{ij}\varepsilon_{kl}(x_{ki}x_{lj} - x_{li}x_{kj}) = x_{\langle kl, \langle ij \rangle},$$

where $1 \leq i \neq j \leq 3$, $1 \leq k \neq l \leq 3$. Here for $k \neq l$, $\langle kl \rangle \in \{1, 2, 3\}$ is such that $\{k, l\} \cup \{\langle kl \rangle\} = \{1, 2, 3\}$, and $\varepsilon_{kl} \in \{\pm 1\}$ is defined by $e_k e_l = \varepsilon_{kl} e_{\langle kl \rangle}$, with $\varepsilon_{kl} = -\varepsilon_{lk}$.

Using the antipode relations (3.3) yield

$$(3.4) \quad \varepsilon_{ij}\varepsilon_{kl}(x_{lj}x_{ki} - x_{li}x_{kj}) = x_{\langle kl, \langle ij \rangle}.$$

Comparing (3.3) and (3.4) we get the relations between commutators

$$[x_{ki}, x_{lj}] = [x_{li}, x_{kj}].$$

Applying the antipode this gives

$$[x_{ki}, x_{lj}] = [x_{kj}, x_{li}],$$

and finally

$$[x_{ki}, x_{lj}] = -[x_{lj}, x_{kj}] = 0,$$

for $k \neq l$ and $i \neq j$. We conclude that $k[C_A]$ is commutative.

Furthermore, the orthogonality of the matrix (x_{ij}) and equations (3.3) for the pairwise commuting elements x_{ij} , give the defining relations for $\mathcal{O}[\mathrm{SO}_3(k)] \simeq \mathcal{O}[\mathrm{PSL}_2(k)]$, therefore determining a unique algebra map

$$(3.5) \quad f : \mathcal{O}[\mathrm{PSL}_2(k)] \rightarrow k[x_{ij} \mid 1 \leq i, j \leq 3] = k[C_A].$$

Because of formula (3.2), this map is a bialgebra map, whence a Hopf algebra map. Thus $k[C_A]$ is isomorphic to $\mathcal{O}[\Gamma]$, where $\Gamma \subseteq \mathrm{PSL}_2(k)$ is an algebraic subgroup, as claimed. \square

3.2. The Hopf algebra $H(n)$. Let $n \geq 2$. Let $H(n)$ be the algebra presented by generators a_{ij}, b_{ij} , $1 \leq i, j \leq n$, and relations

$$A^t B = I = {}^t B A, \quad B^t A = I = {}^t A B,$$

where A and B denote, respectively, the matrices (a_{ij}) and (b_{ij}) , and I is the $n \times n$ identity matrix.

The algebra $H(n)$ has a Hopf algebra structure with multiplication, counit and antipode given, respectively, by

$$\begin{aligned} \Delta(a_{ij}) &= \sum_l a_{il} \otimes a_{lj}, & \epsilon(a_{ij}) &= \delta_{ij}, & \mathcal{S}(a_{ij}) &= b_{ji}, \\ \Delta(b_{ij}) &= \sum_l b_{il} \otimes b_{lj}, & \epsilon(b_{ij}) &= \delta_{ij}, & \mathcal{S}(b_{ij}) &= a_{ji}. \end{aligned}$$

The Hopf algebra $H(n)$ coincides with the universal cosevereign Hopf algebra $H(I_n)$, introduced in [5] (when $k = \mathbb{C}$, it is the Hopf $*$ -algebra associated with Wang's universal compact quantum group in [52]). It is cosemisimple by [8], with $\mathcal{S}^2 = \mathrm{id}$. The n -dimensional vector space V with basis $(e_i)_{1 \leq i \leq n}$ has a natural $H(n)$ -comodule structure $\alpha : V \rightarrow V \otimes H(n)$, where $\alpha(e_i) = \sum_{j=1}^n e_j \otimes a_{ji}$, $1 \leq i \leq n$.

Moreover, for any Hopf algebra H with $\mathcal{S}^2 = \text{id}$, and H -comodule $\alpha : V \rightarrow V \otimes H$, of dimension $\dim V = n$, there exist an $H(n)$ -comodule structure $\alpha' : V \rightarrow V \otimes H(n)$, and a unique Hopf algebra map $f : H(n) \rightarrow H$ such that $(\text{id} \otimes f)\alpha' = \alpha$.

Consider the Hopf subalgebra $H_+(2)$ of $H(2)$ generated, as an algebra, by all products $a_{ij}b_{tl}$, $1 \leq i, j, t, l \leq 2$. The following result is the algebraic generalization of a result proved in [2] (Theorem 5) in the compact quantum group framework. This proof given here is simpler.

Corollary 3.3. *$H_+(2)$ is a commutative Hopf subalgebra of $H(2)$ isomorphic to $\mathcal{O}[\text{PSL}_2(k)]$.*

Proof. Let V be the standard 2-dimensional $H(2)$ -comodule. Keep the notation in the proof of Proposition 3.2. The subalgebra $H_+(2)$ coincides with the subalgebra $k[C_A]$. We have $\mathcal{S}^2 = \text{id}$ in $H(2)$. Consider the surjective Hopf algebra map $f : \mathcal{O}[\text{PSL}_2(k)] \rightarrow H_+(2)$ in (3.5).

On the other hand, there is a natural Hopf algebra inclusion $\mathcal{O}[\text{PSL}_2(k)] \subseteq \mathcal{O}[\text{SL}_2(k)]$. The universal property of $H(2)$ in [5, Theorem 4.2] gives a surjective Hopf algebra map $H(2) \rightarrow \mathcal{O}[\text{SL}_2(k)]$ that induces by restriction a Hopf algebra map $g : H_+(2) \rightarrow \mathcal{O}[\text{PSL}_2(k)]$. The map g defines an inverse of f . Thus f is an isomorphism and we are done. \square

3.3. The finite dimensional case. Let H be a finite dimensional Hopf algebra. Since k has characteristic zero, then H satisfies $\mathcal{S}^2 = \text{id}$ if and only if H is cosemisimple, if and only if H is semisimple [25].

We shall use these facts together with Proposition 3.2 to prove a refinement of [38, Theorem 11] in the finite dimensional context. See Theorem 1.1.

Recall that a finite subgroup of $\text{PSL}_2(k)$, called a *polyhedral group*, is up to isomorphisms, one in following list.

- (i) The cyclic group \mathbb{Z}_n , $n \geq 1$;
- (ii) the dihedral group D_n , of order $2n$, $n \geq 2$;
- (iii) the group T of symmetries of the regular tetrahedra, of order 12;
- (iv) the group O of symmetries of the regular octahedra, of order 24;
- (v) the group I of symmetries of the regular icosahedra, of order 60.

We have $T \simeq \mathbb{A}_4$, $O \simeq \mathbb{S}_4$ and $I \simeq \mathbb{A}_5$.

The Hopf algebra $k^{\mathbb{Z}_n} \simeq k\mathbb{Z}_n$ is cocommutative. For the other subgroups Γ , the coalgebra types of k^Γ are the following:

$$k^{D_n} : \begin{cases} (1, 2; 2, (n-1)/2), & n \text{ odd,} \\ (1, 4; 2, (n/2) - 1), & n \text{ even,} \end{cases}$$

$$k^{\mathbb{A}_4} : (1, 3; 3, 1), \quad k^{\mathbb{S}_4} : (1, 2; 2, 1; 3, 2), \quad k^{\mathbb{A}_5} : (1, 1; 3, 2; 4, 1; 5, 1).$$

Proof of Theorem 1.1. Let V be the simple 2-dimensional comodule corresponding to C . Then $A = V \otimes V^* \simeq M_2(k)$ is an H -comodule algebra. Since H is finite dimensional and cosemisimple, then $\mathcal{S}^2 = \text{id}$. By Remark 3.1, the trace map $\text{Tr} : A \rightarrow k$ is H -colinear.

Applying Proposition 3.2, we get that $k[C_A] \subseteq H$ is a commutative Hopf subalgebra of H isomorphic to k^Γ , where Γ is a finite subgroup of $\text{PSL}_2(k)$. Since $A = V \otimes V^*$ as H -comodules, the subcoalgebra C_A coincides with $\mathcal{CS}(C)$.

It remains to show that Γ is a non cyclic subgroup, which will ensure, in view of the above list, that it has even order. So assume that $\Gamma = \langle g \rangle$, where $g \in \text{PSL}_2(k)$ has finite order. Let $M \in \text{SL}_2(k)$ be a matrix representing g : it follows from the fact that g has finite order that M is diagonalisable, and hence the conjugation action of M on $A = V \otimes V^*$ has at least two trivial subrepresentations, which contradicts the fact that V is a simple H -comodule (since the category of $k[C_A]$ -comodules is a full subcategory of the category of H -comodules).

Consider now the subgroup $G[\chi]$, where $\chi \in C$ is the character of V . Then we see, by examining the decomposition of $V \otimes V^*$ into a direct sum of irreducibles, that $|G[\chi]| = 1, 2$ or 4 . More precisely we have

- (1) $|G[\chi]| = 1 \iff V \otimes V^* \simeq k \oplus W$, where W is a simple k^Γ -comodule of dimension 3.
- (2) $|G[\chi]| = 2 \iff V \otimes V^* \simeq k \oplus kx \oplus Z$, where $x \in G(H)$ has order 2 (in fact $x \in G[\chi]$) and Z is a simple k^Γ -comodule of dimension 2.
- (3) $|G[\chi]| = 4 \iff V \otimes V^* \simeq k \oplus kx \oplus ky \oplus kz$ where $x, y, z \in G(H)$ (and then $G[\chi] = \{1, x, y, z\}$).

The reader will easily check the details (see [38], Theorem 10, for this kind of reasoning). The proof of the last statement of Theorem 1.1 now follows from the coalgebra types of k^Γ , listed above. \square

Remark 3.4. Keep the notation in Theorem 1.1. Assume that $\chi\chi^* = 1 + \lambda$, where λ is an irreducible character of degree 3. This means that possibility (1) in [38, Theorem 11] does not hold.

Then $B = k[\mathcal{CS}(C)]$ coincides with the subalgebra generated by the subcoalgebra C_λ containing λ , of dimension 12, 24 or 60. Further, assume that $\dim k[C_\lambda] = 24$, that is, $k[C_\lambda] \simeq k^{\mathbb{S}_4}$. In view of the fusion rules of representations of \mathbb{S}_4 , $\chi \notin k[C_\lambda]$. The coalgebra types in the remaining cases, imply that $\chi \notin k[C_\lambda]$ for any possible isomorphism class of $k[C_\lambda]$. In particular, $k[C_\lambda]$ is a proper Hopf subalgebra of H .

3.4. A special situation. We use the previous considerations to attach an exact sequence to a cosemisimple Hopf algebra having a faithful comodule of dimension 2, in a special situation. The precise result is the following one.

Recall from [22, 4.1] that the order of an H -comodule V is the smallest natural number n such that $V^{\otimes n}$ contains the trivial H -comodule. For instance, if V is self-dual and nontrivial, then the order of V equals 2.

Theorem 3.5. *Let H be a cosemisimple finite dimensional Hopf algebra having a faithful irreducible comodule V of dimension 2, and assume that $V \otimes V^* \simeq V^* \otimes V$. Then there is a cocentral abelian exact sequence of Hopf algebras*

$$k \rightarrow k^\Gamma \rightarrow H \rightarrow k\mathbb{Z}_m \rightarrow k$$

for a polyedral group Γ of even order and $m \geq 1$, such that m divides the order of V .

Proof. The universal property of $H(2)$ yields a surjective Hopf algebra map $H(2) \rightarrow H$. Denote by H_+ the image of $H_+(2)$ in H . We know from Theorem 1.1 and its proof that $H_+ \simeq k^\Gamma$ for a polyedral group Γ of even order. If C is the coalgebra corresponding to V , we have $H_{\text{coad}} = k[\mathcal{CS}(C)] = H_+$ by Lemma 2.4, and hence the exact sequence (2.6) gives a cocentral exact sequence

$$k \rightarrow k^\Gamma \rightarrow H \rightarrow kM \rightarrow k$$

for a (finite) group M . It remains to check that the group M is cyclic. We have $kM \simeq H/(H_+)^+H$, and hence it is enough to check that $H/(H_+)^+H$ is the group algebra of a cyclic group. For notational simplicity, the elements a_{ij}, b_{ij} of $H(2)$ are still denoted by the same symbol in $H/(H_+)^+H$. In $H/(H_+)^+H$, we have

$$a_{ii}b_{jj} = 1, \quad a_{ij}b_{kl} = 0 \text{ if } i \neq j \text{ or } k \neq l.$$

Hence if $i \neq j$, we have $a_{ij} = a_{ij}a_{ii}b_{jj} = 0$, and the elements a_{ii} are group-like, and they generate M . We have moreover $a_{ii} = a_{ii}b_{jj}a_{jj} = a_{jj}$, and hence M is cyclic.

Let now N be the order of V . Let v_1, v_2 be a basis of V such that $\rho(v_i) = \sum_j v_j \otimes a_{ji}$. Since $V^{\otimes N}$ contains the trivial corepresentation, there exists $v = \sum_{i_1, \dots, i_N} \lambda_{i_1, \dots, i_N} v_{i_1} \otimes \dots \otimes v_{i_N}$ in $V^{\otimes N}$ such that $v \neq 0$ and

$$\rho(v) = \sum_{i_1, \dots, i_N, j_1, \dots, j_N} \lambda_{i_1, \dots, i_N} v_{j_1} \otimes \dots \otimes v_{j_N} \otimes a_{j_1 i_1} \dots a_{j_N i_N} = v \otimes 1.$$

Then, in $V^{\otimes N} \otimes H/(H_+)^+H$, we have

$$v \otimes 1 = \sum_{i_1, \dots, i_N} \lambda_{i_1, \dots, i_N} v_{i_1} \otimes \dots \otimes v_{i_N} \otimes a^N = v \otimes a^N,$$

where $a = a_{i_1 i_1} = \dots = a_{i_N i_N}$. Hence $a^N = 1$, and since m equals the order of a , then m divides N , as claimed. \square

Remark 3.6. The assumption of the previous theorem holds if H is coquasitriangular, or if $V \simeq V^*$. The self-dual situation is studied in the next sections; we have in this case $m \leq 2$.

Corollary 3.7. *Let H and V be as in Theorem 3.5. Then for all irreducible H -module W , the dimension of W divides the order of V .*

Proof. As an algebra, H is a crossed product $k^\Gamma \# k\mathbb{Z}_m$. Then the irreducible representations of H are of the form $W_s := \text{Ind}_{k^\Gamma \otimes kC_s}^H s \otimes U_s$, where s runs over a set of representatives of the orbits of the action of \mathbb{Z}_m in Γ , and $C_s \subseteq \mathbb{Z}_m$ is the stabilizer of s , and U_s is an irreducible representation of C_s . See [32] for a description of the irreducible representations of a general group crossed product. In particular, we have $\dim W_s = [\mathbb{Z}_m : C_s] \dim U_s$, and since $\dim U_s$ divides the order of C_s , then $\dim W_s$ divides m . This implies the corollary since, by Theorem 3.5, m divides the order of V . \square

Remark 3.8. The exact sequence (2.6) is still valid if the cosemisimple Hopf algebra H is not assumed to be finite dimensional, and hence we still have a cocentral exact sequence

$$k \rightarrow \mathcal{O}[G] \rightarrow H \rightarrow kM \rightarrow k$$

where $G \subset \text{PSL}_2(k)$ is an algebraic subgroup and M is a cyclic group. Note however that in the infinite dimensional situation, the Hopf algebra H will not necessarily be a crossed product, because H is not necessarily free as a left (or right) module over its Hopf subalgebra $\mathcal{O}[G]$ [44, 53].

4. HOPF ALGEBRAS WITH A SELF-DUAL FAITHFUL COMODULE

In this section we discuss general results for Hopf algebras having a faithful self-dual comodule.

4.1. The Hopf algebra of a nondegenerate bilinear form. Let $n \geq 2$ and let $E \in \text{GL}_n(k)$. Let $\mathcal{B}(E)$ be the algebra presented by generators a_{ij} , $1 \leq i, j \leq n$, and relations

$$E^{-1} {}^t A E A = I = A E^{-1} {}^t A E,$$

where A denotes the matrix (a_{ij}) and I is the $n \times n$ identity matrix.

There is a Hopf algebra structure on $\mathcal{B}(E)$ with multiplication, counit and antipode given, respectively, by

$$\Delta(a_{ij}) = \sum_l a_{il} \otimes a_{lj}, \quad \epsilon(a_{ij}) = \delta_{ij}, \quad \mathcal{S}(A) = E^{-1} {}^t A E.$$

The Hopf algebra $\mathcal{B}(E)$ was introduced in [12] as the *quantum group of a nondegenerate bilinear form*. The n -dimensional vector space $V = V^E$ with basis $(e_i)_{1 \leq i \leq n}$ has a natural $\mathcal{B}(E)$ -comodule structure $\alpha : V \rightarrow V \otimes \mathcal{B}(E)$, where $\alpha(e_i) = \sum_{j=1}^n e_j \otimes a_{ji}$, $1 \leq i \leq n$. Moreover, the nondegenerate bilinear form $\beta : V \otimes V \rightarrow k$, $\beta(e_i \otimes e_j) = E_{ij}$, is a $\mathcal{B}(E)$ -comodule map.

Moreover, $\mathcal{B}(E)$ is universal with respect to this property. Namely, for any Hopf algebra H and H -comodule V of finite dimension such that $\beta : V \otimes V \rightarrow k$ is a nondegenerate H -invariant bilinear form, then V is a $\mathcal{B}(E)$ -comodule, where $E \in \text{GL}_n(k)$ is any matrix associated to β . In this case, β is a $\mathcal{B}(E)$ -comodule map and there exists a unique Hopf algebra map $f : \mathcal{B}(E) \rightarrow H$ such that $(\text{id} \otimes f)\alpha = \alpha'$, where α and α' are the coactions on V of $\mathcal{B}(E)$ and H , respectively.

For an H -comodule V , the existence of a nondegenerate H -invariant bilinear form on V means exactly that V is self-dual, that is, $V^* \simeq V$ as H -comodules. This is also equivalent to $\mathcal{S}(C) = C$, where $C \subseteq H$ is the subcoalgebra spanned by the matrix coefficients of V .

Isomorphism classes of the Hopf algebras $\mathcal{B}(E)$ are classified by the orbits of the action of $\mathrm{GL}_n(k)$ on itself defined by $G.E = {}^tGEG$, $G, E \in \mathrm{GL}_n(k)$. More precisely, it is shown in [7, Theorem 5.3] that for $E \in \mathrm{GL}_n(k)$, $F \in \mathrm{GL}_m(k)$, the Hopf algebras $\mathcal{B}(E)$ and $\mathcal{B}(F)$ are isomorphic if and only if $n = m$ and E and F are conjugated under the action of $\mathrm{GL}_n(k)$.

Our motivation for studying the Hopf algebras $\mathcal{B}(E)$ is the following basic result.

Proposition 4.1. *Let H be a Hopf algebra having a faithful self-dual H -comodule V .*

- (1) *There exists a surjective Hopf algebra map $\mathcal{B}(E) \rightarrow H$, where the size of the matrix E is the dimension of V .*
- (2) *If H is cosemisimple and $\mathcal{S}^2 = \mathrm{id}$, then the above matrix E is either symmetric or skew-symmetric.*
- (3) *If H is semisimple, then the above matrix E is either symmetric or skew-symmetric.*

Proof. The first assertion follows from the previous discussion and the universal property of $\mathcal{B}(E)$. The second one follows from the Frobenius-Schur theorem for cosemisimple Hopf algebras with $\mathcal{S}^2 = \mathrm{id}$ [6], and the third one follows from the second one and the fact that a semisimple Hopf algebra is automatically cosemisimple and with $\mathcal{S}^2 = \mathrm{id}$ [25] (one may also use directly the Frobenius-Schur theorem for semisimple Hopf algebras from [26]). \square

Remark 4.2. Consider the situation in Proposition 4.1 (3), where V is an irreducible comodule. Then the matrix E is symmetric or skew-symmetric, if and only if the Frobenius-Schur indicator $\nu(V)$ equals 1 or -1 , respectively.

4.2. Exact sequences attached to quotients of $\mathcal{B}(E)$. Let $L \subseteq \mathcal{B}(E)$ denote the linear span of a_{ij} , $1 \leq i, j \leq n$. Let also $\mathcal{B}_+(E) = k[L^2]$ be the subalgebra generated by L^2 , and $\mathcal{B}_-(E)$ be the subspace of odd powers of L .

Then $\mathcal{B}_+(E)$ is a Hopf subalgebra, $\mathcal{B}_-(E)$ is a subcoalgebra, and there is a \mathbb{Z}_2 -algebra grading $\mathcal{B}(E) = \mathcal{B}_+(E) \oplus \mathcal{B}_-(E)$ (the fact that the sum is direct follows from the \mathbb{Z}_2 -action constructed in proof of the next result).

Proposition 4.3. *The Hopf algebra $\mathcal{B}_+(E)$ is a normal Hopf subalgebra of $\mathcal{B}(E)$ and there is a cocentral exact sequence of Hopf algebras*

$$k \rightarrow \mathcal{B}_+(E) \rightarrow \mathcal{B}(E) \rightarrow k\mathbb{Z}_2 \rightarrow k.$$

Proof. It is easy to check the existence of a Hopf algebra map $\zeta : \mathcal{B}(E) \rightarrow k\mathbb{Z}_2$ be defined by $\zeta(a_{ij}) = \delta_{ij}g$, where $1 \neq g \in \mathbb{Z}_2$. The relation $\zeta(x_1) \otimes x_2 =$

$\zeta(x_2) \otimes x_1$ is clearly satisfied for $x = a_{ij}$, $1 \leq i, j \leq n$. Since these generate $\mathcal{B}(E)$, we get that the map ζ is cocentral.

The Hopf algebra map ζ induces a \mathbb{Z}_2 -action on $\mathcal{B}(E)$, given by $g.a_{ij} = -a_{ij}$. It is then clear that $\mathcal{B}_+(E) = \mathcal{B}(E)^{\mathbb{Z}_2} = \mathcal{B}(E)^{\text{co}\zeta}$, and we have our exact sequence by Lemma 2.1. \square

Now let $\mathcal{B}(E) \rightarrow H$ be a Hopf algebra quotient. We denote by H_+ the image of $\mathcal{B}_+(E)$ in H , this is the subalgebra of H generated by the coefficients of $V^{\otimes 2}$.

Proposition 4.4. *Let $\mathcal{B}(E) \rightarrow H$ be a Hopf algebra quotient. Then H_+ is a normal Hopf subalgebra of H , and we have $[H : H_+] = \dim H/(H_+)^+H = 1$ or 2. Moreover if $[H : H_+] = 2$, we have a cocentral short exact sequence of Hopf algebras*

$$k \rightarrow H_+ \rightarrow H \rightarrow k\mathbb{Z}_2 \rightarrow k$$

Proof. It is clear that H_+ is a normal Hopf subalgebra of H . For notational simplicity, the elements a_{ij} of $\mathcal{B}(E)$ are still denoted by the same symbol in any quotient. In $H/(H_+)^+H$, we have

$$a_{ii}a_{jj} = 1, \quad a_{ij}a_{kl} = 0, \quad \text{if } i \neq j \text{ or } k \neq l$$

Hence if $i \neq j$, we have $a_{ij} = a_{ii}^2 a_{ij} = 0$, and $a_{ii} = a_{ii}a_{jj}^2 = a_{jj}$. It follows that $H/(H_+)^+H \simeq k$ or $H/(H_+)^+H \simeq k\mathbb{Z}_2$, depending whether $g = a_{ii} = 1$ or not. If $H/(H_+)^+H \simeq k\mathbb{Z}_2$, the canonical projection $H \rightarrow H/(H_+)^+H$ is defined by the same formula as the map ζ in the proof of the previous proposition, and one concludes in a similar manner. \square

Remark 4.5. We gave the direct (easy) proofs for the exact sequences in the previous propositions. These exact sequences are special cases of (2.6).

4.3. \mathbb{Z}_2 -grading on comodule categories and adjoint subcategory.

By [7, Theorem 1.1] there is an equivalence of tensor categories

$$(4.1) \quad \mathcal{B}(E) - \text{comod} \simeq \mathcal{O}_q[\text{SL}_2(k)] - \text{comod},$$

where q is a root of the polynomial $X^2 + \text{Tr}(E^t E^{-1})X + 1$. The equivalence sends the standard $\mathcal{B}(E)$ -comodule V^E to the standard 2-dimensional $\mathcal{O}_q[\text{SL}_2(k)]$ -comodule U .

We shall assume in this subsection that the roots of the polynomial $X^2 + \text{Tr}(E^t E^{-1})X + 1$ are generic. In this case recall that $\mathcal{O}_q[\text{SL}_2(k)]$ -comod is a semisimple tensor category whose simple objects are represented by the comodules $(U_n)_{n \geq 0}$, where $\dim U_n = n+1$ (in particular, $U_0 = k$ and $U_1 = U$ is the standard 2-dimensional comodule). These obey the following fusion rules:

$$(4.2) \quad U_i \otimes U_j \simeq U_{i+j} \oplus U_{i+j-2} \oplus \cdots \oplus U_{|i-j|},$$

for all $i, j \geq 0$.

Let $\mathcal{C} = \mathcal{O}_q[\text{SL}_2(k)] - \text{comod}$. Consider the full abelian subcategories \mathcal{C}_+ , respectively \mathcal{C}_- , whose simple objects are $(U_i)_{i \text{ even}}$, respectively $(U_i)_{i \text{ odd}}$.

The fusion rules (4.2) imply that $\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-$ is a faithful \mathbb{Z}_2 -grading on \mathcal{C} . In particular, \mathcal{C}_+ is a tensor subcategory of \mathcal{C} . See [41, p. 7].

Lemma 4.6. $\mathcal{C}_+ = \mathcal{C}_{\text{ad}}$ is the tensor subcategory of \mathcal{C} generated by U_2 .

In particular, since $U_2 \otimes U_2 \simeq U_4 \oplus U_2 \oplus U_0$, we have $(\mathcal{C}_{\text{ad}})_{\text{ad}} = \mathcal{C}_{\text{ad}}$.

Proof. It is clear. \square

Let \mathcal{C}^E denote the category of finite dimensional $\mathcal{B}(E)$ -comodules. The equivalence (4.1) has the following corollary.

Corollary 4.7. *There is a faithful \mathbb{Z}_2 -grading $\mathcal{C}^E = \mathcal{C}_+^E \oplus \mathcal{C}_-^E$, where $\mathcal{C}_+^E = \mathcal{C}_{\text{ad}}^E$ is the full tensor subcategory generated by the simple comodule V_2^E . Moreover $\mathcal{B}_\pm(E) - \text{comod} = \mathcal{C}_\pm^E$.*

Proof. The first assertion follows from the equivalence (4.1). The adjoint category $\mathcal{C}_+^E = \mathcal{C}_{\text{ad}}^E$ is the category of corepresentations of a Hopf subalgebra B of $\mathcal{B}(E)$. Indeed, it follows from the first assertion that B is the subalgebra generated by the matrix coefficients of the simple comodule V_2^E . In view of the relation $V^E \otimes V^E \simeq k \oplus V_2^E$, B coincides with the subalgebra of $\mathcal{B}(E)$ generated by the products $a_{ij}a_{kl}$, $1 \leq i, j, k, l \leq n$; that is, $B = \mathcal{B}_+(E)$. Similarly, $\mathcal{B}_-(E) - \text{comod} = \mathcal{C}_-^E$. \square

In the next proposition we shall consider the one-parameter deformation of $\text{SO}_3(k)$ as studied by Takeuchi in [50]. See also [16, 42].

We use here the notation $\mathcal{O}_{q^2}[\text{SO}_3(k)]$ to indicate the Hopf algebra denoted $\mathcal{A}_{q^2}(3)$ in [50].

Proposition 4.8. *There is an isomorphism of Hopf algebras $\mathcal{O}_q[\text{SL}_2(k)]_+ \simeq \mathcal{O}_{q^2}[\text{SO}_3(k)]$. Therefore the equivalence (4.1) restricts to an equivalence of tensor categories*

$$(4.3) \quad \mathcal{B}_+(E) - \text{comod} \simeq \mathcal{O}_{q^2}[\text{SO}_3(k)] - \text{comod}.$$

Recall that in (4.1) q is a root of the polynomial $X^2 + \text{Tr}(E^t E^{-1})X + 1$, and we are assuming that q is generic.

Proof. Note that $\mathcal{O}_q[\text{SL}_2(k)] = \mathcal{B}(E_q)$, where E_q is the 2×2 -matrix defined by $(E_q)_{11} = (E_q)_{22} = 0$, $(E_q)_{12} = -q^{-1/2}$, $(E_q)_{21} = q^{1/2}$. So that, denoting by x_{ij} , $1 \leq i, j \leq 2$, the generators of $\mathcal{O}_q[\text{SL}_2(k)]$, we have that $\mathcal{O}_q[\text{SL}_2(k)]_+ = \mathcal{B}_+(E_q)$ is the Hopf subalgebra generated by the products $x_{ij}x_{tl}$, $1 \leq i, j, t, l \leq 2$.

Since the parameter q is generic by assumption, there is an injective Hopf algebra map $f : \mathcal{O}_{q^2}[\text{SO}_3(k)] \rightarrow \mathcal{O}_q[\text{SL}_2(k)]$ [50, Proposition 8], [11]. Furthermore, in view of the definition of f in [50], the image of f coincides with $\mathcal{O}_q[\text{SL}_2(k)]_+ = \mathcal{B}_+(E_q)$, therefore defining an isomorphism $\mathcal{O}_q[\text{SL}_2(k)]_+ \simeq \mathcal{O}_{q^2}[\text{SO}_3(k)]$.

Finally, note that the equivalence (4.1), being an equivalence of tensor categories, must preserve the adjoint subcategories. Then it induces by

restriction the equivalence (4.3). This finishes the proof of the proposition. \square

Proposition 4.9. *Let \mathcal{C} be the tensor category of finite dimensional H -comodules, where H is a semisimple Hopf algebra having a self-dual faithful simple comodule V . Then there is a \mathbb{Z}_2 -grading $\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-$, where $\mathcal{C}_+ = \mathcal{C}_{\text{ad}}$. Moreover, \mathcal{C}_+ is the full tensor subcategory generated by the simple constituents of $V^{\otimes 2}$ and $\mathcal{C}_+ = H_+ - \text{comod}$.*

Proof. We have a surjective Hopf algebra map $f : \mathcal{B}(E) \rightarrow H$. This induces a dominant tensor functor $F : \mathcal{C}^E \rightarrow \mathcal{C}$. Therefore, in view of Corollary 4.7, $\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-$ is a \mathbb{Z}_2 -grading on \mathcal{C} , where $\mathcal{C}_\pm = F(\mathcal{C}_\pm^E)$. Since $F(V^E) = V$, then it is clear that \mathcal{C}_+ is generated by the simple constituents of $V^{\otimes 2}$, whence, by definition, $\mathcal{C}_+ = \mathcal{C}_{\text{ad}} = H_+ - \text{comod}$. \square

Remark 4.10. Consider the dominant tensor functor $F : \mathcal{C}^E \rightarrow \mathcal{C}$ in the proof of Proposition 4.9. In view of the equivalences (4.1) and (4.3), F gives rise to dominant tensor functors

$$\mathcal{O}_q[\text{SL}_2(k)] - \text{comod} \rightarrow H - \text{comod}, \quad \mathcal{O}_{q^2}[\text{SO}_3(k)] - \text{comod} \rightarrow H_+ - \text{comod},$$

for an appropriate choice of the (generic) parameter $q \in k^\times$.

5. THE 2×2 CASE

5.1. Classification of $\mathcal{B}(E)$'s. In the 2×2 case, the orbits of the $\text{GL}_2(k)$ -action are represented by one the matrices

$$E_{\mathfrak{m}} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_q = \begin{pmatrix} 0 & -q^{-\frac{1}{2}} \\ q^{\frac{1}{2}} & 0 \end{pmatrix}, \quad q \in k \setminus \{0, 1\}.$$

Moreover, the matrices E_q and $E_{q'}$ belong to the same orbit if and only if $q' = q^{\pm 1}$.

Hence in this case the Hopf algebra $\mathcal{B}(E)$ is isomorphic to exactly one of the Hopf algebras in the following list:

- (1) $\mathcal{O}_q[\text{SL}_2(k)]$, $q \in k \setminus \{0, 1\}$, corresponding to the matrices E_q ;
- (2) $\mathcal{O}[\text{SL}_2(k)]$, corresponding to the matrix E_1 ;
- (3) $\mathcal{B}(E_{\mathfrak{m}})$.

See [7, Theorem 5.3], [12, Section 6]. Note that, by [7, Theorem 1.1], the category of comodules over the Hopf algebra $\mathcal{B}(E_{\mathfrak{m}})$ is tensor equivalent to the category of $\mathcal{O}[\text{SL}_2(k)]$ -comodules.

In particular, we have the following

Corollary 5.1. *Suppose that the matrix E is symmetric (respectively, skew-symmetric). Then the Hopf algebra $\mathcal{B}(E)$ is isomorphic to $\mathcal{O}_{-1}[\text{SL}_2(k)]$ (respectively, to $\mathcal{O}[\text{SL}_2(k)]$). \square*

Example 5.2. Let H be a finite dimensional Hopf algebra and let $C \subseteq H$ be a simple subcoalgebra such that $\dim C = 4$ and $\mathcal{S}(C) = C$. Suppose that the Hopf subalgebra $k[C]$ is not semisimple. Then the order of $S^2|_C$ is an integer $m > 1$ [25]. The assumptions imply that there is a Hopf algebra map $\mathcal{B}(E) \rightarrow H$, whose image is $k[C]$, for some $E \in \mathrm{GL}_2(k)$. Since $k[C]$ is not cosemisimple, then $\mathcal{B}(E) \simeq \mathcal{O}_q(\mathrm{SL}_2(k))$, for some root of unity $q \in k^\times$, $q \neq \pm 1$.

This fact was proved by Stefan in [46, Theorem 1.5], and then it was used in the classification of Hopf algebras of low dimension [46, 35]. Note that when q is a root of unity of odd order, $\mathcal{O}_q(\mathrm{SL}_2(k))$ contains a central Hopf subalgebra $A \simeq \mathcal{O}(\mathrm{SL}_2(k))$. This gives a central Hopf subalgebra $\bar{A} \subseteq k[C]$, such that the quotient $H/H\bar{A}^+$ is dual to a pointed Hopf algebra.

The following corollary gives the classification of cosemisimple Hopf algebras satisfying condition (4) in the Nichols-Richmond Theorem [38] under the assumption that the simple 2-dimensional comodule is self-dual and faithful.

Corollary 5.3. *Let H be a cosemisimple Hopf algebra. Suppose H has a family $\{C_n : n \geq 1\}$ of simple subcoalgebras such that $\dim C_n = n^2$, satisfying, for all $n \geq 2$,*

$$C_n \mathcal{S}(C_2) = C_{n-1} + C_{n+1}, \quad n \text{ even}, \quad C_n C_2 = C_{n-1} + C_{n+1}, \quad n \text{ odd}.$$

Assume in addition that $H = k[C]$, where $C = C_1$, and $\mathcal{S}(C) = C$. Then H is isomorphic to one of the Hopf algebras $\mathcal{O}_{\pm 1}[\mathrm{SL}_2(k)]$, $\mathcal{B}(E_m)$, or $\mathcal{O}_q[\mathrm{SL}_2(k)]$, $q \in k \setminus \{0, \pm 1\}$ not a root of unity.

Proof. Let $\mathcal{G}(H^*)$ denote the Grothendieck ring of the category of finite dimensional H -comodules. The assumption $H = k[C]$ implies that $\mathcal{G}(H^*)$ is spanned by the irreducible characters χ_i corresponding to the simple subcoalgebras C_i , $i \geq 1$. The assumption also implies that there is a surjective Hopf algebra map $f : \mathcal{B}(E) \rightarrow H$, for some $E \in \mathrm{GL}_2(k)$, such that $f(\bar{C}) = C$, where $\bar{C} \subseteq \mathcal{B}(E)$ is the simple subcoalgebra corresponding to the standard 2-dimensional corepresentation V of $\mathcal{B}(E)$. Furthermore, f induces an isomorphism between the Grothendieck rings. Then f is an isomorphism, by [7, Theorem 1.2]. The result follows from the classification of the Hopf algebras $\mathcal{B}(E)$ in the 2×2 case. \square

5.2. The Hopf algebra $\mathcal{O}_{-1}[\mathrm{SL}_2(k)]$. Recall from [24] the definition of the quantum groups $\mathcal{O}_q[\mathrm{SL}_2(k)]$, where $q \in k$, $q \neq 0$. We shall be interested in the case $q = -1$.

As an algebra, $\mathcal{O}_{-1}[\mathrm{SL}_2(k)]$ is presented by generators a, b, c and d , with relations

$$(5.1) \quad \begin{aligned} bc &= cb, & ad &= da, \\ ba &= -ab, & ca &= -ac, & db &= -bd, & dc &= -cd, \\ ad + bc &= 1. \end{aligned}$$

We shall use the notation $\mathcal{A} = \mathcal{O}_{-1}[\mathrm{SL}_2(k)]$. Similarly, A will denote the commutative Hopf algebra $\mathcal{O}[\mathrm{SL}_2(k)]$, with standard commuting generators x, y, z, t and defining relation $xz - yt = 1$.

The i, j coefficient of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (respectively, $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$) will be alternatively denoted by a_{ij} (respectively, x_{ij}), $1 \leq i, j \leq 2$.

The algebra \mathcal{A} has a Hopf algebra structure with comultiplication, counit and antipode determined by the formulas

$$(5.2) \quad \Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}, \quad \epsilon(a_{ij}) = \delta_{ij}, \quad \mathcal{S} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

The following lemma is well-known. See [43, Proposition 2.2].

Lemma 5.4. *Let V be an irreducible representation of \mathcal{A} . Then the dimension of V equals 1 or 2.* \square

Consider the \mathbb{Z}_2 -algebra grading $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ as in Subsection 4.3. The Hopf subalgebra \mathcal{A}_+ is the largest commutative Hopf subalgebra of \mathcal{A} [51, Theorem 15.4].

In the same way, we have a \mathbb{Z}_2 -algebra grading $A = A_+ \oplus A_-$. By [51, Corollary 15.3], $A_+ \simeq \mathcal{A}_+$ as Hopf algebras.

The category $\mathrm{comod} - \mathcal{A}_+$ corresponds to the tensor subcategory of A -comodules having only even weights [51, Theorem 15.5]. As Hopf algebras, A_+ is isomorphic to the algebra of coordinate functions on the group $\mathrm{PSL}_2(k) \simeq \mathrm{SO}_3(k)$. Note, however, that A is *not* free as a left (or right) module over its Hopf subalgebra A_+ [44, 53].

There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -algebra grading on \mathcal{A} (respectively, on A), such that a, d (respectively, x, t) have degree $(1, 0)$, and b, c (respectively, y, z) have degree $(0, 1)$. Associated to this grading, there is a deformed product \bullet in A , defined in the form

$$u \bullet v = (-1)^{j(k+l)} uv,$$

for homogeneous elements $u, v \in A$, of degrees (i, j) and (k, l) , respectively. See [51, Section 15].

This defines an associative product which makes (A, \bullet) into a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra with the same unit as A . Moreover, there is an isomorphism of graded algebras $\lambda : (A, \bullet) \rightarrow \mathcal{A}$, determined by $\lambda(x_{ij}) = a_{ij}$, $1 \leq i, j \leq 2$. The vector space isomorphism underlying λ is furthermore a coalgebra isomorphism [51, Proposition 15.1].

We have $\lambda(A_{\pm}) = \mathcal{A}_{\pm}$. Moreover, the product in $A_+ \subseteq A$ coincides with the deformed product \bullet , since A_+ sits in degrees $(0, 0)$ and $(1, 1)$. Hence the restriction of λ induces a canonical Hopf algebra isomorphism $A_+ \rightarrow \mathcal{A}_+$.

Recall that, by Proposition 4.3, \mathcal{A}_+ is a normal Hopf subalgebra of \mathcal{A} , and we have a short exact sequence of Hopf algebras $k \rightarrow \mathcal{A}_+ \rightarrow \mathcal{A} \xrightarrow{\zeta} k\mathbb{Z}_2 \rightarrow k$, where the map $\zeta : \mathcal{A} \rightarrow k\mathbb{Z}_2$ is determined by $\zeta(a_{ij}) = \delta_{ij}g$, $1 \neq g \in \mathbb{Z}_2$.

Proposition 5.5. λ induces a commutative diagram with exact rows

$$(5.3) \quad \begin{array}{ccccccccc} k & \longrightarrow & A_+ & \longrightarrow & A & \xrightarrow{\zeta'} & k\mathbb{Z}_2 & \longrightarrow & k \\ \downarrow & & \downarrow \simeq & & \downarrow \lambda & & \downarrow = & & \downarrow \\ k & \longrightarrow & \mathcal{A}_+ & \longrightarrow & \mathcal{A} & \xrightarrow{\zeta} & k\mathbb{Z}_2 & \longrightarrow & k. \end{array}$$

Proof. The proof is straightforward. \square

Remark 5.6. The \mathbb{Z}_2 -gradings $A = A_+ \oplus A_-$ and $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ are associated to the \mathbb{Z}_2 -coaction arising from the Hopf algebra surjection $\zeta' : A \rightarrow k\mathbb{Z}_2$, respectively, $\zeta : \mathcal{A} \rightarrow k\mathbb{Z}_2$. Moreover, $\lambda : A \rightarrow \mathcal{A}$ is a \mathbb{Z}_2 -graded map.

On the other hand, we have $A_+ = A_{(0,0)} \oplus A_{(1,1)}$, $A_- = A_{(1,0)} \oplus A_{(0,1)}$ and similarly, $\mathcal{A}_+ = \mathcal{A}_{(0,0)} \oplus \mathcal{A}_{(1,1)}$, $\mathcal{A}_- = \mathcal{A}_{(1,0)} \oplus \mathcal{A}_{(0,1)}$. This can be seen using that the set

$$\{a^i b^j c^k : 0 \leq i, j, k\} \cup \{b^i c^j d^k : 0 \leq i, j, 0 < k\},$$

forms a basis of the algebra \mathcal{A} , and similarly for A . See [24].

Suppose $\mathcal{J} \subseteq \mathcal{A}$ is a \mathbb{Z}_2 -homogeneous ideal of \mathcal{A} , that is, \mathcal{J} is an ideal and $\mathcal{J} = \mathcal{J}_+ \oplus \mathcal{J}_-$, where $\mathcal{J}_\pm = \mathcal{J} \cap \mathcal{A}_\pm$. Then it follows from the definition of the product \cdot that (\mathcal{J}, \cdot) is an ideal of (\mathcal{A}, \cdot) . Therefore, $J = \lambda^{-1}(\mathcal{J})$ is a \mathbb{Z}_2 -homogeneous ideal of A .

5.3. Finite quantum subgroups of $\mathcal{O}_{-1}[\mathrm{SL}_2(k)]$. Let H be a nontrivial (i.e. non commutative) finite dimensional quotient Hopf algebra of $\mathcal{A} = \mathcal{O}_{-1}[\mathrm{SL}_2(k)]$. In particular, H is semisimple and cosemisimple.

Proposition 5.7. *The Hopf algebra H fits into an abelian cocentral exact sequence*

$$(5.4) \quad k \rightarrow k^\Gamma \rightarrow H \rightarrow k\mathbb{Z}_2 \rightarrow k,$$

where Γ is a finite subgroup of even order of $\mathrm{PSL}_2(k)$. The adjoint Hopf subalgebra H_{coad} is commutative and isomorphic to k^Γ .

Proof. Since H is not commutative and H_+ is, the exact sequence follows from Proposition 4.4. The Hopf algebra H_{coad} is determined by $H_{\mathrm{coad}} - \mathrm{comod} = (H - \mathrm{comod})_{\mathrm{ad}}$, and we have $(H - \mathrm{comod})_{\mathrm{ad}} = H_+ - \mathrm{comod}$ by Proposition 4.9. Hence $H_{\mathrm{coad}} \cong H_+ \cong k^\Gamma$. \square

Remark 5.8. Another way to prove the proposition is to use Theorem 3.5.

Since the extension (5.4) is cocentral, the proposition implies that H^* has a central group-like element of order 2.

Remark 5.9. Consider another quotient Hopf algebra $\mathcal{A} \rightarrow H'$, giving rise to an exact sequence $k \rightarrow k^{\Gamma'} \rightarrow H' \rightarrow k\mathbb{Z}_2 \rightarrow k$, as in Proposition 5.7. Assume that $H \simeq H'$ as Hopf algebras. Then $\Gamma \simeq \Gamma'$ and this exact sequence is isomorphic to (5.4).

Proof. By construction, we have $k^\Gamma = H_{\text{coad}}$ and $k^{\Gamma'} = H'_{\text{coad}}$. Therefore a Hopf algebra isomorphism $H \rightarrow H'$ must restrict to an isomorphism $k^\Gamma \rightarrow k^{\Gamma'}$. Then it induces an isomorphism of the corresponding exact sequences. \square

As explained in Subsection 2.1, the extension (5.4) is determined by an action of \mathbb{Z}_2 on Γ , or in other words, by a group automorphism θ of order 2, and a pair (σ, τ) of compatible cocycles.

By Remark 5.9, this data determines the isomorphism class of H which is that of a bicrossed product $k^{\Gamma\tau} \#_\sigma k\mathbb{Z}_2$.

Remark 5.10. Since \mathbb{Z}_2 is cyclic, we may apply [34, Lemma 1.2.5] to the dual extension $k^{\mathbb{Z}_2\sigma} \#_\tau k\Gamma$, whence we get that the class of σ is trivial in $H^2(\mathbb{Z}_2, (k^\Gamma)^\times)$.

By [29, Proposition 5.2], after eventually changing the representative of the class of τ in $H^2(\Gamma, (k^{\mathbb{Z}_2})^\times)$, we may assume that $\sigma = 1$. This can be seen alternatively, applying [27, Theorem 4.4].

The automorphism θ and the class of τ will be explicitly determined in Lemmas 5.17 and 5.15, respectively.

Recall that a finite Hopf algebra over the field of complex numbers is called a *Kac algebra* if it is a C^* -algebra and all structure maps are C^* -algebra maps.

Corollary 5.11. *Suppose $k = \mathbb{C}$. Then H admits a Kac algebra structure.*

Proof. It follows from Proposition 5.7 and [28]. \square

Let H be as in (5.4). The next corollary describes the category $\text{Rep } H$ of finite dimensional representations of H . Let Vec^Γ denote the fusion category of finite dimensional Γ -graded vector spaces.

Corollary 5.12. *There is an action of \mathbb{Z}_2 on Vec^Γ by tensor autoequivalences such that $\text{Rep } H$ is equivalent to the \mathbb{Z}_2 -equivariantization $(\text{Vec}^\Gamma)^{\mathbb{Z}_2}$.*

Proof. It follows from Proposition 5.7 and [37, Proposition 3.5]. \square

Remark 5.13. For every finite subgroup Γ of $\text{PSL}_2(k)$, as listed in (i)–(v) of Subsection 3.3, there is a subgroup $\tilde{\Gamma}$ (often denoted 2Γ) of $\text{SL}_2(k)$ such that $\tilde{\Gamma}/\{\pm I\} = \Gamma$.

The groups $\tilde{\Gamma}$ are called the *binary polyhedral groups*. These groups are non-split central extensions of \mathbb{Z}_2 by the corresponding polyhedral group:

$$(5.5) \quad 0 \rightarrow \mathbb{Z}_2 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1.$$

Furthermore, we have $Z(\tilde{\Gamma}) \simeq \mathbb{Z}_2$. A binary dihedral 2-group is also called a generalized quaternion group. There are isomorphisms $\tilde{T} \simeq \text{SL}(2, 3)$, $\tilde{I} \simeq \text{SL}(2, 5)$ [9], [48, Theorem 6.17].

We shall denote by $\omega \in H^2(\Gamma, \mathbb{Z}_2)$ the nontrivial cohomology class arising from the central extension (5.5).

Remark 5.14. For the finite subgroups of $\mathrm{PSL}_2(k)$ listed before, we have $H^2(\mathbb{Z}_n, k^\times) = 1$, $H^2(D_n, k^\times) = 1$, for n odd, $H^2(D_n, k^\times) \simeq \mathbb{Z}_2$, for n even, while the groups $H^2(T, k^\times)$, $H^2(O, k^\times)$ and $H^2(I, k^\times)$ are all cyclic of order 2. See [23, Proposition 4.6.4 and Theorems 4.8.3, 4.8.5].

In particular, if Γ is one of the groups T , O , I or D_n , n even, then $\tilde{\Gamma}$ is a representation group of Γ . That is, every (irreducible) projective representation of Γ lifts uniquely to an (irreducible) linear representation of $\tilde{\Gamma}$. See [23, Chapter 3].

5.4. Description. Again H is a nontrivial finite dimensional quotient Hopf algebra of $\mathcal{A} = \mathcal{O}_{-1}[\mathrm{SL}_2(k)]$. We shall next determine the cohomological data arising from the extension (5.4).

Recall that such an extension is determined by a group automorphism $\theta \in \mathrm{Aut} \Gamma$ of order 2, and an element of the group $\mathrm{Opext}(k^\Gamma, k\mathbb{Z}_2)$ associated to θ .

Let $\alpha_{ij} = \pi(a_{ij})$, $1 \leq i, j \leq 2$, denote the image in H of the standard generators of \mathcal{A} . Thus the α_{ij} 's span the simple subcoalgebra C of H such that $H = k[C]$.

Let also $p : H \rightarrow k\mathbb{Z}_2$ denote the (cocentral) Hopf algebra map with $k^\Gamma \simeq H^{\mathrm{cop}} \subseteq H$, and $p(a_{ij}) = \delta_{ij}x$, where $1 \neq x \in \mathbb{Z}_2$.

Lemma 5.15. *There is an isomorphism of coalgebras $H \rightarrow k^{\tilde{\Gamma}}$ preserving the Hopf subalgebra k^Γ .*

Identifying $\mathbb{Z}_2 \simeq \widehat{\mathbb{Z}_2}$, the class of the cocycle $\tau : \Gamma \times \Gamma \rightarrow (k^{\mathbb{Z}_2})^\times$ arising from the exact sequence (5.4) is determined by

$$(5.6) \quad \tau(s, t)(p) = p \circ \omega(s, t),$$

for all $p \in \widehat{\mathbb{Z}_2}$, $s, t \in \Gamma$, where $\omega \in H^2(\Gamma, \mathbb{Z}_2)$ is as in Remark 5.13.

Proof. Since H is nontrivial, the commutative diagram (5.3) can be extended to a commutative diagram with exact rows

$$(5.7) \quad \begin{array}{ccccccccc} k & \longrightarrow & \mathcal{A}_+ & \longrightarrow & A & \longrightarrow & k\mathbb{Z}_2 & \longrightarrow & k \\ \downarrow & & \downarrow = & & \downarrow \lambda & & \downarrow = & & \downarrow \\ k & \longrightarrow & \mathcal{A}_+ & \longrightarrow & \mathcal{A} & \longrightarrow & k\mathbb{Z}_2 & \longrightarrow & k \\ \downarrow & & \downarrow \pi & & \downarrow \pi & & \downarrow = & & \downarrow \\ k & \longrightarrow & k^\Gamma & \longrightarrow & H & \longrightarrow & k\mathbb{Z}_2 & \longrightarrow & k, \end{array}$$

where commutativity of the bottom right square follows from the construction of the exact sequence in the proof of Proposition 4.4. In particular, π is a \mathbb{Z}_2 -graded map with respect to the canonical \mathbb{Z}_2 -gradings on \mathcal{A} and H .

Letting $\mathcal{J} \subseteq \mathcal{A}$ denote the kernel of π , it follows that \mathcal{J} is a \mathbb{Z}_2 -homogeneous ideal. Then, by Remark 5.6, $\lambda^{-1}(\mathcal{J}) = J$ is an ideal of A . Moreover, J is also a coideal of A , since λ is a coalgebra isomorphism.

Since the composition $\pi\lambda : A \rightarrow H$ is surjective, this allows us to identify $H \simeq k^S$ as coalgebras, where $S \subseteq \mathrm{SL}_2(k)$ is a finite subset. By the above, $\lambda^{-1}(\mathcal{J})$ is a Hopf ideal of A . Then k^S is a quotient Hopf algebra of A , that is, S is a subgroup of $\mathrm{SL}_2(k)$. Commutativity of the diagram (5.7) implies that S is isomorphic to the binary polyhedral group associated to Γ .

Since $S \simeq \tilde{\Gamma}$, then S fits into an exact sequence (5.5), and thus S can be identified with $\mathbb{Z}_2 \times \Gamma$ with the product given, for $s, t \in \Gamma$, $x, y \in \mathbb{Z}_2$, by the formula

$$(x, s)(y, t) = (xy\omega_0(s, t), st),$$

where $\omega_0 \in Z^2(\Gamma, \mathbb{Z}_2)$ represents the class ω . Then k^S is identified with $k^{\mathbb{Z}_2 \times \Gamma} \simeq k^\Gamma \otimes k^{\mathbb{Z}_2}$ with the (dual) coproduct:

$$\Delta(e_g \otimes p)(x \otimes s \otimes y \otimes t) = (e_g \otimes p)(xy\omega_0(s, t) \otimes st) = e_g(st)p(xy\omega_0(s, t)),$$

for all $s, t, g \in \Gamma$, $x, y \in \mathbb{Z}_2$, $p \in k^{\mathbb{Z}_2}$. Then, for all $p \in \widehat{\mathbb{Z}}_2 \subseteq k^{\mathbb{Z}_2}$, we get

$$\Delta(e_g \otimes p) = \sum_{st=g} (p \circ \omega_0)(s, t)(e_s \otimes p) \otimes (e_t \otimes p).$$

Comparing this expression with formula (2.3), we obtain formula (5.6) for τ . This finishes the proof of the lemma. \square

Remark 5.16. In view of Lemma 5.15, the Hopf algebras H can be regarded as 'deformations' of the binary polyhedral groups $\tilde{\Gamma}$. Observe that, since every Sylow subgroup of $\tilde{\Gamma}$ is either generalized quaternion or cyclic, then $\tilde{\Gamma}$ does not admit cocycle deformations.

Indeed, by the classification of (dual) cocycle deformations in [33, 10], such deformations are classified by pairs (S, α) , where S is a subgroup and α is a nondegenerate 2-cocycle on S . In particular, the order of S is necessarily a square. For the reason we mentioned, $\tilde{\Gamma}$ contains no nontrivial such subgroup which admits a nondegenerate 2-cocycle.

Let $\rho : \Gamma \rightarrow \mathrm{GL}(V)$ be a finite dimensional projective representation of Γ on the vector space V . Recall that a factor set of ρ is a 2-cocycle $\alpha : \Gamma \times \Gamma \rightarrow k^\times$ such that

$$\rho(s)\rho(t) = \alpha(s, t)\rho(st),$$

for all $s, t \in \Gamma$. Under linear equivalence, the classes of projective representations with a given factor set α correspond to isomorphism classes of representations of the twisted group algebra $k_\alpha\Gamma$.

In particular, if ρ, ρ' are projective representations on the vector space V , and $X, X' \subseteq k^\Gamma$ are the subspaces of matrix coefficients of ρ and ρ' , respectively, then ρ is linearly equivalent to ρ' if and only if $X = X'$.

In contrast with the notion of linear equivalence, there is a notion of projective equivalence. Projectively equivalent representations may give rise to factor sets differing by a coboundary. They correspond to isomorphic (linear) representations, when lifted to the representation group.

Let τ be the 2-cocycle determined by (5.6). That is, $\tau_p \in Z^2(\Gamma, k^\times)$ is a 2-cocycle representing the class of $p \circ \omega$, where $1 \neq p \in \widehat{\mathbb{Z}}_2$. Since $|H^2(\Gamma, k^\times)| \leq 2$, we may and shall assume in what follows that $\tau_p^{-1} = \tau_p$.

Suppose ρ is a projective representation of Γ with factor set τ_p , that lifts to a self-dual representation of the group $\widetilde{\Gamma}$, that is, such that ρ^* is projectively equivalent to ρ . Since $\tau_p^{-1} = \tau_p$, then ρ^* is indeed linearly equivalent to ρ .

In particular, if $X \subseteq k^\Gamma$ is the subspace of matrix coefficients of ρ , then we have $\mathcal{S}(X) = X$.

Lemma 5.17. *Suppose that the quotient Hopf algebra $\mathcal{O}_{-1}[\mathrm{SL}_2(k)] \rightarrow H$, where $H \simeq k^{\Gamma\tau} \# k\mathbb{Z}_2$, affords the automorphism $\theta \in \mathrm{Aut}(\Gamma)$ of order 2. Then there exists a 2-dimensional faithful irreducible projective representation ρ of Γ with factor set τ_p such that $\rho \circ \theta$ is linearly equivalent to ρ .*

Conversely, given such ρ and θ , there exists a unique noncommutative semisimple Hopf algebra extension $H = k^{\Gamma\tau} \# k\mathbb{Z}_2$ affording the automorphism θ , such that H is a quotient of $\mathcal{O}_{-1}[\mathrm{SL}_2(k)]$.

Proof. We shall identify $\mathbb{Z}_2 \simeq \widehat{\mathbb{Z}}_2$ in what follows. Let $H \simeq k^{\Gamma\tau} \# k\mathbb{Z}_2$, affording the automorphism $\theta \in \mathrm{Aut}(\Gamma)$ of order 2. Since H is a quotient $\mathcal{O}_{-1}[\mathrm{SL}_2(k)] \rightarrow H$, there exists a 4-dimensional simple subcoalgebra $C \subseteq H$ such that $k[C] = H$ and $\mathcal{S}(C) = C$.

In view of Remark 2.2, $C = X \# p$, where X is identified with a 4-dimensional simple subcoalgebra of the twisted group algebra $(k_{\tau_p}\Gamma)^*$. Let ρ be the irreducible projective representation corresponding to X . In view of Lemma 5.15, under the canonical coalgebra isomorphism $H \rightarrow k^{\widetilde{\Gamma}}$, C corresponds to a self-dual faithful irreducible representation of $\widetilde{\Gamma}$ that lifts the projective representation ρ . Thus ρ is a faithful self-dual projective representation. Moreover, since $\tau_p = \tau_p^{-1}$, then ρ^* is linearly equivalent to ρ ; that is, $\mathcal{S}(X) = X$ in k^Γ .

Recall that the action of $p \in \mathbb{Z}_2$ on Γ is given by the automorphism θ . In view of formula (2.4) for the antipode of H , condition $\mathcal{S}(C) = C$ implies that $p.X = \mathcal{S}(X) = X$, where $\cdot : k\mathbb{Z}_2 \otimes k^\Gamma \rightarrow k^\Gamma$ is the action transpose to \triangleleft . This amounts to the condition that $\rho \circ \theta$ be linearly equivalent to ρ .

Indeed, $p.X$ is the span of matrix coefficients of the projective representation $\rho \circ \theta$ of Γ . On the other hand, if $c = x \# p \in C$, then, in view of formula (2.4) for the antipode of H , we have

$$\begin{aligned} \mathcal{S}(c) &= \sum_s \tau_p(s^{-1}, s)^{-1} x(s) e_{(s \triangleleft p)^{-1}} \# p \\ &= \sum_s \tau_p((s \triangleleft p)^{-1}, s \triangleleft p)^{-1} x(s \triangleleft p) e_{s^{-1}} \# p, \end{aligned}$$

and the last expression belongs to $p.\mathcal{S}(X) \# p = p.X \# p$. Hence $\mathcal{S}(C) = C$, if and only if, $p.X = X$, if and only if, $\rho \circ \theta$ is linearly equivalent to ρ .

Conversely, suppose given such θ and ρ . Let $1 \neq p \in \widehat{\mathbb{Z}}_2$. We have in particular, $\theta^*(\tau_p) = \tau_p^{-1}$. This implies that $(1, \tau)$ is a pair of compatible cocycles, thus giving rise to a Hopf algebra $H = k^{\Gamma\tau} \# k\mathbb{Z}_2$.

Let $X \subseteq k^{\Gamma}$ be the span of the matrix coefficients of the 2-dimensional self-dual faithful irreducible projective representation ρ , and let $C = X \# p \subseteq H$.

By assumption, $\rho \circ \theta$ is linearly equivalent to ρ . Since the action of $p \in \mathbb{Z}_2$ on Γ is given by the automorphism θ , this implies that $p.X = X$, where $\cdot : k\mathbb{Z}_2 \otimes k^{\Gamma} \rightarrow k^{\Gamma}$ is the action transpose to \triangleleft . Hence, as before, formula (2.4) for the antipode of H implies that $\mathcal{S}(C) = C$. Similarly, formula (2.2) for the multiplication implies that $k[C] = H$. Since H is not commutative, then H must be a quotient of $\mathcal{O}_{-1}[\mathrm{SL}_2(k)]$. This finishes the proof of the lemma. \square

Let us recall from [31] some properties of the nontrivial Hopf algebras \mathcal{A}_{4n} and \mathcal{B}_{4n} constructed there.

Consider the presentation of the dihedral group D_n by generators s_+ and s_- satisfying the relations

$$s_{\pm}^2 = (s_+s_-)^n = 1.$$

Both $H = \mathcal{A}_{4n}$ and $H = \mathcal{B}_{4n}$ are extensions $k \rightarrow k^{\mathbb{Z}_2} \rightarrow H \rightarrow kD_n \rightarrow k$ associated to the matched pair $(k^{\mathbb{Z}_2}, kD_n)$, where the action of \mathbb{Z}_2 on D_n is given by

$$(5.8) \quad a \triangleright s_{\pm} = s_{\mp}.$$

We shall use the notation $\mathcal{A}[\widetilde{D}_n] := \mathcal{A}_{4n}$ and $\mathcal{B}[\widetilde{D}_n] := \mathcal{B}_{4n}$. Then $\mathcal{A}[\widetilde{D}_n]$ and $\mathcal{B}[\widetilde{D}_n]$ are nonequivalent representatives of classes in $\mathrm{Opext}(k^{\mathbb{Z}_2}, kD_n)$ associated to the action (5.8). Moreover, every extension arising from this matched pair is isomorphic to one of $\mathcal{A}[\widetilde{D}_n]$ or $\mathcal{B}[\widetilde{D}_n]$.

Suppose that Γ is one of the groups $T \simeq \mathbb{A}_4$, $O \simeq \mathbb{S}_4$, or $I \simeq \mathbb{A}_5$.

In the first two cases we have $\mathrm{Aut} \Gamma \simeq \mathbb{S}_4$, while $\mathrm{Aut} \mathbb{A}_5 \simeq \mathbb{S}_5$. In all cases, the automorphisms θ of order 2 of Γ are induced by the adjoint action of a transposition $(\cdot\cdot)$, or a product of two disjoint transpositions $(\cdot\cdot)(\cdot\cdot)$, viewed as elements in the corresponding symmetric group.

Consider the case where $\Gamma = \mathbb{A}_4$. In this case, $k_{\tau_p}\Gamma$ has exactly 3 irreducible representations of degree 2, and only one of them is self-dual: this can be seen from the fact that $\widehat{\mathbb{A}}_4$ is a representation group of \mathbb{A}_4 . See [47, Table A.12] for the character table of $\widetilde{\mathbb{A}}_4$, where the self-dual representation is the one with character χ_3 .

Then, in this case, ρ corresponds to χ_3 and it is necessarily stable under all automorphisms θ of \mathbb{A}_4 .

Similarly, when $\Gamma = \mathbb{S}_4$, every automorphism is inner, and therefore stabilizes all projective representations.

By Lemma 5.17, in the cases $\Gamma = \mathbb{A}_4$ or \mathbb{S}_4 , we have two nonisomorphic associated Hopf algebras with a faithful self-dual comodule of dimension 2.

We shall denote the Hopf algebra H associated to θ by $\mathcal{A}[\widetilde{\Gamma}]$ or $\mathcal{B}[\widetilde{\Gamma}]$, if θ corresponds to a transposition or to a product of two disjoint transpositions, respectively.

Now assume that $\Gamma = \mathbb{A}_5$. It follows from inspection of the character table of $\widetilde{\mathbb{A}}_5$ [47, Table A.19] that $k_{\tau_p}\mathbb{A}_5$ has 2 nonequivalent irreducible representations of degree 2, one of degree 4 and one of degree 6. Moreover, we have

Lemma 5.18. *The nonequivalent projective representations of degree 2 are conjugated by an outer automorphism of \mathbb{A}_5 induced by a transposition in \mathbb{S}_5 .*

Proof. Consider the presentation of $\widetilde{\mathbb{A}}_5$ by generators a and b and relations $a^5 = b^3 = (ab)^2 = -1$. There are in $\widetilde{\mathbb{A}}_5$ nine classes under conjugation represented, respectively, by $1, -1, a, a^2, a^3, a^4, b, b^2$ and ab . According to [47, Table A.19], $\widetilde{\mathbb{A}}_5$ has two nonequivalent irreducible representations of degree 2, whose characters χ and χ' are determined, respectively, by the following table:

	1	-1	a	a^2	a^3	a^4	b	b^2	ab
χ	2	-2	φ^+	$-\varphi^-$	φ^-	$-\varphi^+$	1	-1	0
χ'	2	-2	φ^-	$-\varphi^+$	φ^+	$-\varphi^-$	1	-1	0

where $\varphi^\pm = \frac{1 \pm \sqrt{5}}{2}$.

Let $\rho, \rho' : \widetilde{\mathbb{A}}_5 \rightarrow \mathrm{SL}_2(k)$ be the irreducible representations with characters χ, χ' . Consider the surjective group homomorphism $\pi : \widetilde{\mathbb{A}}_5 \rightarrow \mathbb{A}_5$ with kernel $\{\pm 1\}$, such that $\pi(a) = (12345)$ and $\pi(b) = (153)$.

Letting \bar{s} denote the element $\pi(s)$, $s \in \mathbb{A}_5$, the classes of \mathbb{A}_5 under conjugation are represented by $\bar{1} = 1, \bar{a} = (12345), \bar{a}^3, \bar{b}$ and $\bar{a}\bar{b}$. We have also that the class of \bar{a}^3 coincides with the class of (21345) .

After composing with a suitable section $\mathbb{A}_5 \rightarrow \widetilde{\mathbb{A}}_5$ of π , ρ and ρ' define irreducible projective representations $\bar{\rho}$ and $\bar{\rho}'$ of \mathbb{A}_5 , respectively. The projective characters are determined by the following table:

	1	(12345)	(21345)	(153)	(12)(34)
$\bar{\chi}$	2	φ^+	φ^-	1	0
$\bar{\chi}'$	2	φ^-	φ^+	1	0

In particular, $\bar{\rho}$ and $\bar{\rho}'$ are not equivalent. Furthermore, if θ is the automorphism determined by the adjoint action of the transposition (12), then we see from the table above, that $\bar{\chi}' = \bar{\chi} \circ \theta$. This implies the statement on the projective representations in view of [23, Theorem 7.1.11]. \square

In view of the above lemma and Lemma 5.17, when the automorphism θ corresponds to a transposition in \mathbb{S}_5 , the bicrossed product associated to θ has no irreducible self-dual comodule of dimension 2. When θ corresponds

to a product of two disjoint transpositions, the automorphism is inner and we have an associated Hopf algebra that we shall denote $\mathcal{B}[\widetilde{\mathbb{A}}_5]$, as before.

Theorem 5.19. *Let H be a nontrivial Hopf algebra quotient of $\mathcal{O}_{-1}[\mathrm{SL}_2(k)]$. Then H is isomorphic to exactly one of the Hopf algebras $\mathcal{B}[\widetilde{T}]$, $\mathcal{A}[\widetilde{\Gamma}]$ or $\mathcal{B}[\widetilde{\Gamma}]$, where $I \simeq \mathbb{A}_5$, and Γ is one of the groups D_n , $n \geq 2$, $T \simeq \mathbb{A}_4$, $O \simeq \mathbb{S}_4$.*

Proof. By Proposition 5.7 and Theorem 5.19, since H is nontrivial, then H is an extension (5.4). Hence H corresponds to the class of some pair (σ, τ) in $\mathrm{Opext}(k^\Gamma, k\mathbb{Z}_2)$. By Lemma 5.15, we know that τ_p represents the class $p \circ \omega$. Moreover, by [34, Lemma 1.2.5], [27, Theorem 4.4], we may assume that $\sigma = 1$. Moreover, by Remark 5.9, two quotients are isomorphic if and only if the exact sequences are isomorphic.

Consider the case where Γ is one of the groups T , O or I . In view of Lemma 5.17 and the previous discussion, it remains to determine which possible automorphisms θ, θ' , give isomorphic Hopf algebras H, H' . Since a Hopf algebra isomorphism $H \rightarrow H'$ is an isomorphism of extensions, this is the case if and only if the associated matched pairs are isomorphic, which amounts to the automorphisms θ and θ' being conjugated in $\mathrm{Aut}(\Gamma)$.

The case where $\Gamma = D_n$, $n \geq 2$, will be treated separately in Lemma 5.20. This will imply, as a byproduct, a comparison with the nontrivial examples studied by Masuoka in [31]. \square

The following facts about fusion rules for the irreducible characters of k^{D_n} , $n \geq 2$, will be used in the proof of our next lemma.

If n is odd, k^{D_n} has two irreducible characters 1, a , of degree 1, and $r = (n - 1)/2$ irreducible characters of degree 2, χ_1, \dots, χ_r , such that

$$(5.9) \quad a\chi_i = \chi_i, \quad \chi_i\chi_j = \begin{cases} \chi_{i+j} + \chi_{|i-j|}, & i + j \leq r, \\ \chi_{n-(i+j)} + \chi_{|i-j|}, & i + j > r, \end{cases}$$

where $\chi_0 = 1 + a$.

On the other hand, if n is even, k^{D_n} has four irreducible characters 1, a , b , $c = ab$, of degree 1, and $r = n/2 - 1$ irreducible characters of degree 2, χ_1, \dots, χ_r , such that

$$(5.10) \quad a\chi_i = \chi_i, \quad b\chi_i = \chi_{n/2-i}, \quad \chi_i\chi_j = \begin{cases} \chi_{i+j} + \chi_{|i-j|}, & i + j \leq n/2, \\ \chi_{n-(i+j)} + \chi_{|i-j|}, & i + j > n/2, \end{cases}$$

where $\chi_0 = 1 + a$, $\chi_{n/2} = b + c$.

Lemma 5.20. *Let H be a nontrivial Hopf algebra quotient of $\mathcal{O}_{-1}[\mathrm{SL}_2(k)]$ such that H fits into an exact sequence (5.4), with $\Gamma = D_n$, $n \geq 2$. Then H is isomorphic to one of the Hopf algebras $\mathcal{A}[\widetilde{D}_n]$ or $\mathcal{B}[\widetilde{D}_n]$.*

By [31, Proposition 3.13] the Hopf algebras $\mathcal{A}[\widetilde{D}_n]$ and $\mathcal{B}[\widetilde{D}_n]$ are not isomorphic, if $n \geq 3$.

Note that the fusion rules for $\mathcal{A}[\tilde{D}_n]$ or $\mathcal{B}[\tilde{D}_n]$ are the same as those for $k^{D_{2n}}$, as given by (5.10) [31, Proposition 3.9]. Then $\mathcal{A}[\tilde{D}_n]$, $\mathcal{B}[\tilde{D}_n]$ do satisfy the assumptions in the lemma.

Proof. In view of previous classification results, we shall assume in the proof that $n \neq 2$. That is, we shall not consider the case $\dim H = 8$, where the result is well-known.

Formula (2.3) for the comultiplication in H implies that $H = (k\Gamma)^* \oplus (k_{\tau_p}\Gamma)^*$, where $k_{\tau_p}\Gamma$ is the twisted group algebra, and $1 \neq p \in \mathbb{Z}_2$. In particular, every irreducible H -comodule has dimension 1 or 2. See [23, Theorem 3.7.3].

Let $\chi \in C$ be the irreducible character of C . Then, decomposing $\chi\chi^* = \chi^2$ into a sum of irreducible characters, we get $|G[\chi]| = 2$. Indeed, the assumption that $k[C] = H$ together with $\dim H > 8$, imply that $|G[\chi]| \neq 4$.

Let $G[\chi] = \{1, a\}$, where $a^2 = 1$. Then $\chi^2 = 1 + a + \lambda$, where λ is an irreducible character of degree 2, such that $a \in G[\lambda]$. Thus $k[\lambda] = H_{\text{coad}} \simeq k^{D_n}$. In particular, $kG[\chi]$ is normal in H_{coad} . We claim that $kG[\chi]$ is a central Hopf subalgebra of H .

Since $a\lambda = \lambda = \lambda a$, and the simple subcoalgebra containing λ generates H_{coad} , then the quotient Hopf algebra $H_{\text{coad}}/H_{\text{coad}}(kG[\chi])^+$ is cocommutative, by [36, Remark 3.2.7 and Corollary 3.3.2].

This implies that $a\mu = \mu = \mu a$, for all irreducible character $\mu \in H_{\text{coad}}$ of degree 2. By assumption, $\dim H > 8$. and taking into account the fusion rules for the irreducible characters in k^{D_n} in (5.9) and (5.10), we may assume that $a \in k^{D_n}$ is the only nontrivial group-like element with this property.

Indeed, otherwise, we would have n even and $b\chi_1 = \chi_1$, implying that $n = 4$. Then $\dim H = 16$ and $|G(H)| = 4$, thus the lemma follows in this case from [21].

Hence a must be stable under the action of \mathbb{Z}_2 coming from the sequence (5.4), because this is an action by Hopf algebra automorphisms. Hence $kG[\chi]$ is a normal (therefore central) Hopf subalgebra of H , as claimed.

Moreover, since $aC = C = Ca$, and C generates H , then as before $H/H(kG[\chi])^+$ is cocommutative. Therefore we get an exact sequence

$$(5.11) \quad k \rightarrow k^{\mathbb{Z}_2} \rightarrow H \rightarrow kF \rightarrow k,$$

where F is a group of order $2n$.

We shall show next that $F \simeq D_n$ and the action $\mathbb{Z}_2 \times F \rightarrow F$ associated to the exact sequence (5.11) coincides with (5.8). This will imply the lemma, in view of [31, Proposition 3.11].

Denote by $\pi : H \rightarrow kF$ the projection in (5.11). We may decompose $\pi(\chi) = x + y$, where $x, y \in F$. Moreover, since χ is a faithful character, then x and y generate F .

Since $\chi^* = \chi$, then either $x^2 = y^2 = 1$ or $x = y^{-1}$. If the last possibility holds, then $F = \langle x, y \rangle$ would be cyclic, thus implying that H is commutative, against the assumption. Therefore $x^2 = y^2 = 1$ in F .

On the other hand, we have $\chi^2 = 1 + a + \lambda$, and $\pi(\chi^2) = x^2 + y^2 + xy + yx = 2.1 + xy + (xy)^{-1}$. Hence $\pi(\lambda) = xy + (xy)^{-1}$. But the components of $\pi(\lambda)$ generate a subgroup of index 2 of F whose group algebra is isomorphic to $\pi(H_{\text{coad}}) = \pi(k^{D_n})$. Hence $\langle xy \rangle \simeq \mathbb{Z}_n$. Thus we obtain the relation $(xy)^n = 1$. Hence $F \simeq D_n$.

As in the proof in [31, pp. 209], the simple subcoalgebra C is a left $k\langle a \rangle$ -submodule, and its image in kF is spanned by the group-likes x and y . Then C is a crossed product $C = k^{\mathbb{Z}_2} \# k\{x, y\}$, and therefore the set $\{x, y\}$ is an orbit under the action of \mathbb{Z}_2 . Letting $s_+ = x$, $s_- = y$, we recover the action (5.8). This finishes the proof of the lemma. \square

The following lemma gives the degrees of the irreducible representations of H .

Lemma 5.21. *Let p be the order of the subgroup $\Gamma^{\mathbb{Z}_2}$ of fixed points of Γ under the action of \mathbb{Z}_2 . Then H has $2p$ irreducible representations of degree 1 and the remaining ones are of degree 2. In particular, $|G(H^*)| \geq 4$.*

Proof. As an algebra, H is a smash product $k^\Gamma \# k\mathbb{Z}_2$. Then the irreducible representations of H are of the form $W_s := \text{Ind}_{k^\Gamma \otimes kC_s}^H s \otimes U_s$, where s runs over a set of representatives of the orbits of g in Γ , $C_s \subseteq \mathbb{Z}_2$ is the stabilizer of s , and U_s is an irreducible representation of C_s [32].

In particular, we have $\dim W_s = [\mathbb{Z}_2 : C_s]$. Hence $\dim W_s = 1$ if and only if $s \in \Gamma^{\mathbb{Z}_2}$. Note in addition that such an automorphism g must have a nontrivial fixed point (otherwise, Γ would be abelian of odd order); whence the claimed inequality for $|G(H^*)|$. \square

Remark 5.22. Let $H = \mathcal{A}[\tilde{\Gamma}]$ or $\mathcal{B}[\tilde{\Gamma}]$, be one of the nontrivial Hopf algebras corresponding to $\tilde{\Gamma}$.

Let $p \geq 2$ be the order of the subgroup of invariants in Γ under the action of \mathbb{Z}_2 , and let $q = (\dim H - 2p)/4$. Then, as an algebra, H is of type $(1, 2p; 2, q)$. See Lemma 5.21. Explicitly, we have the following algebra types, for each possible isomorphism class:

$$\begin{aligned} \mathcal{A}[\tilde{D}_n] &: (1, 4; 2, n-1), & \mathcal{B}[\tilde{D}_n] &: (1, 4; 2, n-1), & (n \geq 2) \\ \mathcal{A}[\tilde{T}] &: (1, 4; 2, 5), & \mathcal{B}[\tilde{T}] &: (1, 8; 2, 4), \\ \mathcal{A}[\tilde{O}] &: (1, 8; 2, 10), & \mathcal{B}[\tilde{O}] &: (1, 16; 2, 8), \\ \mathcal{B}[\tilde{I}] &: (1, 8; 2, 28). \end{aligned}$$

On the other hand, by Lemma 5.15, $H \simeq k^{\tilde{\Gamma}}$ as coalgebras. Hence the coalgebra types are the following:

$$\mathcal{A}[\tilde{D}_n], \mathcal{B}[\tilde{D}_n], n \geq 2 : (1, 4; 2, n-1),$$

$$\mathcal{A}[\tilde{T}], \mathcal{B}[\tilde{T}] : (1, 3; 2, 3; 3, 1),$$

$$\mathcal{A}[\tilde{O}], \mathcal{B}[\tilde{O}] : (1, 2; 2, 3; 3, 2; 4, 1),$$

$$\mathcal{B}[\tilde{I}] : (1, 1; 2, 2; 3, 2; 4, 2; 5, 1; 6, 1).$$

See for instance [47, A.5].

Remark 5.23. Let H be one of the Hopf algebra $\mathcal{A}[\tilde{\Gamma}]$ or $\mathcal{B}[\tilde{\Gamma}]$. Then the fusion rules of H are the same as the ones of $\tilde{\Gamma}$.

Proof. The result is known if $\tilde{\Gamma} = \tilde{D}_n$, see [31]. Assume now that $\tilde{\Gamma} = \tilde{T}$, \tilde{O} or \tilde{I} . It is enough, to describe the fusion rules of H , to describe the product of irreducible characters in H . These are, using the previous notation, of the form $\chi \# 1$, $\chi' \# p$, where χ is an ordinary irreducible character of Γ and χ' is an irreducible projective character of Γ . In view of the product formula 2.2, we need to describe the action of θ on (projective) characters. If θ is inner (for $\mathcal{B}[\tilde{T}]$, $\mathcal{A}[\tilde{O}]$, $\mathcal{B}[\tilde{O}]$, $\mathcal{B}[\tilde{I}]$) this action is trivial and hence the fusion rules are unchanged.

If $H = \mathcal{A}[\tilde{T}]$, the automorphism θ permutes the characters $\chi_2 - \chi_3$ and $\chi_4 - \chi_5$ in Table A.12 of [47]. However the computation of the fusion rules of \tilde{T} show that after these permutations, the fusion rules of $\mathcal{A}[\tilde{\Gamma}]$ remain the same. \square

Proof of Theorem 1.2. Let H be a semisimple Hopf algebra containing a simple subcoalgebra C of dimension 4 such that $\mathcal{S}(C) = C$. Let $k[C] \subseteq H$ be the Hopf subalgebra generated by C .

By Proposition 4.1 and Corollary 5.1, there is a surjective Hopf algebra map $\mathcal{H} \rightarrow k[C]$, where $\mathcal{H} = \mathcal{O}[\mathrm{SL}_2(k)]$, if $\nu(V) = -1$, and $\mathcal{H} = \mathcal{O}_{-1}[\mathrm{SL}_2(k)]$, if $\nu(V) = 1$. Then the proof of the theorem follows from Theorem 5.19. \square

6. APPLICATIONS

In this section we shall give some applications of the main results in previous sections to the classification of semisimple Hopf algebras.

6.1. Semisimple Hopf algebras with character degrees at most 2.

Recall that if C is a simple subcoalgebra of a semisimple Hopf algebra, and $\chi \in C$ is the irreducible character contained in C , then $G[\chi]$ is the subgroup of the group-like elements g such that $g\chi = \chi$, or equivalently, such that g appears with positive multiplicity (necessarily equal to 1) in the product $\chi\chi^*$. We shall also denote $B[\chi] = k[\mathcal{CS}(C)]$.

We give for completeness the proof of the following lemma that will be used later on. See [36, Lemma 2.4.1].

Lemma 6.1. *Let H be a semisimple Hopf algebra and let $\chi, \chi' \in H$ be irreducible characters. Then the following are equivalent:*

- (i) *The product $\chi^*\chi'$ is irreducible.*
- (ii) *For all irreducible character $\lambda \neq 1$, $m(\lambda, \chi\chi^*) = 0$ or $m(\lambda, \chi'(\chi')^*) = 0$.*

Here $m(\lambda, \chi\chi^*)$, $m(\lambda, \chi'(\chi')^*)$, denote the multiplicity of λ in $\chi\chi^*$ and in $\chi'(\chi')^*$, respectively.

Proof. Let $\zeta = \chi^*\chi'$. Then ζ is irreducible if and only if $m(1, \zeta\zeta^*) = 1$. On the other hand,

$$\zeta\zeta^* = \chi^*\chi'(\chi')^*\chi = \chi^*\chi + \sum_{\mu \neq 1} m(\mu, \chi'(\chi')^*)\chi^*\mu\chi.$$

Therefore, $m(1, \zeta\zeta^*) = 1$ if and only if for all $\mu \neq 1$, with $m(\mu, \chi'(\chi')^*) > 0$, we have $m(1, \chi^*\mu\chi) = 0$ or equivalently, $m(\mu, \chi\chi^*) = 0$. \square

Let H be a semisimple Hopf algebra such that $\deg \chi \leq 2$, for all irreducible character $\chi \in H$. Let $B \subseteq H$ denote the adjoint Hopf subalgebra $B := H_{\text{coad}}$. Hence $B = k[B[\chi] : \deg \chi = 2]$ is generated as an algebra by the Hopf subalgebras $B[\chi]$.

By Theorem 1.1 $B[\chi]$ is commutative and, in view of the coalgebra structure of H , isomorphic to $k^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ or k^{D_n} , $n \geq 3$, for all irreducible character χ of degree 2. In addition, $G[\chi] \neq 1$, for all such χ .

Lemma 6.2. *Suppose χ is an irreducible character of H of degree 2, such that $|G[\chi]| = 2$. Then $G[\chi] \subseteq \bigcap_{\deg \chi'=2} B[\chi'] \subseteq Z(B)$.*

Proof. Since $\bigcap_{\deg \chi'=2} B[\chi'] \subseteq B[\chi']$, for all χ' of degree 2, then it is a central Hopf subalgebra of B , because the $B[\chi']$'s are commutative and generate B as an algebra. Thus it will be enough to show that $G[\chi] \subseteq \bigcap_{\deg \chi'=2} B[\chi']$.

Let $\chi' \in H$ be any irreducible character of degree 2. Then the product $\chi^*\chi'$ cannot be irreducible, because H has no irreducible character of degree 4. In view of Lemma 6.1, there is an irreducible character $\lambda \neq 1$ that appears with positive multiplicity both in $\chi\chi^*$ and in $\chi'(\chi')^*$. This means, in particular, that the Hopf subalgebra $k[\lambda]$, generated by the simple subcoalgebra containing λ , is contained in $B[\chi] \cap B[\chi']$.

If $\deg(\lambda) = 1$, then $1 \neq \lambda \in G[\chi] \cap G[\chi']$. Since $G[\chi]$ is of order 2 by assumption, this implies that $G[\chi] \subseteq G[\chi'] \subseteq B[\chi']$.

If $\deg(\lambda) > 1$, then λ is irreducible of degree 2. In this case we have decompositions $\chi\chi^* = 1 + g + \lambda$ and $\chi'(\chi')^* = 1 + g' + \lambda$, where $G[\chi] = \{1, g\}$ and $G[\chi'] = \{1, g'\}$.

Since $g\chi = \chi$, then also $g\lambda = \lambda$, that is, $G[\chi] \subseteq G[\lambda]$. Therefore $G[\chi] \subseteq G[\lambda] \subseteq k[\lambda] \subseteq B[\chi']$. \square

Corollary 6.3. *Let H be a semisimple Hopf algebra such that $\deg \chi \leq 2$, for all irreducible character $\chi \in H$. Let $B = H_{\text{coad}}$ be the adjoint Hopf subalgebra. Then one of the following conditions holds:*

(i) B has a central Hopf subalgebra isomorphic to $k\Gamma$, where

$$\Gamma \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2, \\ \text{\scriptsize } m \text{ times}$$

such that $B/B(k\Gamma)^+$ is cocommutative, where $1 \leq m \leq n$, and n is the number of irreducible characters χ of H such that $|G[\chi]| = 2$, or

(ii) B is cocommutative.

Further, if (ii) holds and H is not cocommutative, then 4 divides $\dim B$ and $G(B)$ is generated by elements of order 2.

Proof. We may assume H is not cocommutative. If H has an irreducible character χ of degree 2 such that $|G[\chi]| = 2$, then, by Lemma 6.2, $G[\chi] \subseteq Z(B)$. Let Γ be the subgroup generated by the groups $G[\chi]$ such that $|G[\chi]| = 2$. Thus the group algebra of Γ is a Hopf subalgebra contained in the center of B .

Further, it follows also from Lemma 6.2 that $\Gamma \cap B[\chi] = \Gamma \cap G[\chi] \neq 1$, for all irreducible character χ with $|G[\chi]| = 4$. Then, [36, Remark 3.2.7 and Corollary 3.3.2], the quotient Hopf algebra $B/B(k\Gamma)^+$ is cocommutative, and (i) holds in this case.

Otherwise, $|G[\chi]| = 4$, for all irreducible character χ of degree 2. Since $kG[\chi] \subseteq B$, for all χ , then 4 divides $\dim B$. On the other hand, by dimension, $\chi\chi^* = \sum_{g \in G[\chi]} g \in kG(H)$, for all irreducible character χ of degree 2. Then $B \subseteq kG(H)$ is cocommutative.

Moreover, by Theorem 1.1, we have in this case $G[\chi] \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, for all irreducible χ of degree 2. Then (ii) holds, since these subgroups generate $G(B)$. \square

Theorem 6.4. *Let H be a nontrivial semisimple Hopf algebra such that $\deg \chi \leq 2$, for all irreducible character $\chi \in H$. Then H is not simple. More precisely, let $\Gamma \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ be the subgroup of $G(H)$ in Corollary 6.3 (i). Then one of the following possibilities holds:*

- (i) *There is a central abelian exact sequence $k \rightarrow k\Gamma \rightarrow H \rightarrow kF \rightarrow k$, where F is a finite group, or*
- (ii) *There is a cocentral exact sequence $k \rightarrow B \rightarrow H \rightarrow kU(\mathcal{C}) \rightarrow k$, where $B = H_{\text{coad}} \subsetneq H$, and $U(\mathcal{C})$ is the universal grading group of $\mathcal{C} = H - \text{comod}$.*

Moreover, in case (ii), either B is commutative or it contains a central Hopf subalgebra isomorphic to $k\Gamma$.

In particular, either H or H^* contains a nontrivial central group-like element.

Proof. Suppose first that $H_{\text{coad}} = B \subsetneq H$. Then (ii) holds in view of the properties of the adjoint Hopf subalgebra B . See Subsection 2.3.

If, on the other hand, $B = H$, then by Corollary 6.3 we have (i). \square

Let H be a semisimple Hopf algebra. Recall from [32] that H is called *lower semisolvable* if there exists a chain of Hopf subalgebras $H_{n+1} = k \subseteq$

$H_n \subseteq \cdots \subseteq H_1 = H$ such that H_{i+1} is a normal Hopf subalgebra of H_i , for all i , and all factors $\overline{H}_i := H_{i+1}/H_{i+1}H_i^+$ are commutative or cocommutative.

Dually, H is called *upper semisolvable* if there exists a chain of quotient Hopf algebras $H_{(0)} = H \rightarrow H_{(1)} \rightarrow \cdots \rightarrow H_{(n)} = k$ such that each of the maps $H_{(i-1)} \rightarrow H_{(i)}$ is normal, and all factors $H_i := H_{(i-1)}^{\text{copi}}$ are commutative or cocommutative.

We have that H is upper semisolvable if and only if H^* is lower semisolvable [32].

Remark 6.5. Suppose that H fits into an exact sequence $k \rightarrow K \rightarrow H \rightarrow \overline{H} \rightarrow k$. If K is lower semisolvable and \overline{H} is commutative or cocommutative, then H is lower semisolvable. On the other hand, if K is commutative or cocommutative and \overline{H} is upper semisolvable, then H is upper semisolvable.

Corollary 6.6. *Let H be a semisimple Hopf algebra such that $\deg \chi \leq 2$, for all irreducible character $\chi \in H$. Then H is lower semisolvable.*

Proof. We may assume H is not trivial. Then one of the possibilities (i) or (ii) in Theorem 6.4 holds. If (i) holds, then H is both upper and lower semisolvable.

Suppose that (ii) holds. Since in this case B is a proper Hopf subalgebra, so in particular its irreducible comodules have degree at most 2, we may assume inductively that B is lower semisolvable. Then H is lower semisolvable as well. The corollary is thus proven. \square

Remark 6.7. It follows from Corollary 6.6 and [32] that a semisimple Hopf algebra H with character degrees of irreducible comodules at most 2 have the Frobenius property: that is, the dimensions of the irreducible H -modules divide the dimension of H .

As a consequence of the above results, we have the following proposition. This is proved by Izumi and Kosaki in the context of Kac algebras [19, Corollary IX.9], and under an additional restriction in [36, Theorem 4.6.5].

Proposition 6.8. *Suppose H is of type $(1, 2; 2, n)$ as a coalgebra. Then H is commutative.*

Proof. If $H \neq H_{\text{coad}}$, then H fits into an exact sequence

$$(6.1) \quad k \rightarrow B \rightarrow H \rightarrow kU(\mathcal{C}) \rightarrow k,$$

where $B = H_{\text{coad}}$. We may assume inductively that B is commutative, thus the extension (6.1) is abelian. Hence (6.1) induces an action of $U(\mathcal{C})$ on B by Hopf algebra automorphisms. Moreover, since $G(B) = G(H)$ is of order 2, then $U(\mathcal{C})$ acts trivially on $G(B)$. Then $g \in Z(H)$, because H is a crossed product with respect to this action.

If $H = H_{\text{coad}}$, then by Theorem 6.4, we have a central abelian exact sequence $k \rightarrow kG(H) \rightarrow H \rightarrow kF \rightarrow k$, where F is a finite group. In any case we get that $G(H) \subseteq Z(H)$. Then the proposition follows from [36, Corollary 4.6.8]. \square

6.2. Coalgebra type of a simple Hopf algebra of dimension 60. There are three known examples of nontrivial semisimple Hopf algebras which are simple as Hopf algebras in dimension 60. Two of them are given by the Hopf algebras \mathcal{A}_0 and $\mathcal{A}_1 \simeq \mathcal{A}_0^*$ constructed by Nikshych [40]. We have $\mathcal{A}_0 = (k\mathbb{A}_5)^J$, where $J \in k\mathbb{A}_5 \otimes k\mathbb{A}_5$ is an invertible twist lifted from a subgroup isomorphic to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$.

A third example is given by the self-dual Hopf algebra \mathcal{B} constructed in [17]. In this case $\mathcal{B} = (kD_3 \otimes kD_5)^J$, where J is an invertible twist lifted from a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In this subsection we show that \mathcal{A}_1 and \mathcal{B} are the only nontrivial simple Hopf algebras of dimension 60 with their respective coalgebra type. See Proposition 6.10 and Corollary 6.12.

Remark 6.9. The coalgebra types of these examples are the following:

$$(6.2) \quad \mathcal{A}_1 : (1, 1; 3, 2; 4, 1; 5, 1), \quad \mathcal{A}_0 : (1, 12; 4, 3), \quad \mathcal{B} : (1, 4; 2, 6; 4, 2),$$

and we have $G(\mathcal{A}_0) \simeq \mathbb{A}_4$, $G(\mathcal{B}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. The coalgebra type of \mathcal{A}_1 is that of $k^{\mathbb{A}_5}$. That of \mathcal{B} was computed in [17, 4.2]. The proof of the statement for $\mathcal{A}_0 = \mathcal{A}_1^*$ is given during the proof of Lemma 6.8 in [4], using the classification of irreducible representations in twisting of group algebras [13]. \square

In what follows H will be a semisimple Hopf algebra of dimension 60.

Proposition 6.10. *Suppose $|G(H)| = 1$. Then $H \simeq k^{\mathbb{A}_5}$ or $H \simeq \mathcal{A}_1$.*

Proof. By [36] every semisimple Hopf algebra of dimension < 60 has a nontrivial group-like element. Therefore, if $|G(H)| = 1$, then H can contain no proper Hopf subalgebra. The proposition follows from [15, Corollary 9.14]. \square

Suppose $G(H) \neq 1$ and H has an irreducible character χ of degree 2. Let $C \subseteq H$ be the simple subcoalgebra containing χ , and consider the Hopf subalgebra $B =: B[\chi] = k[\mathcal{CS}(C)] \subseteq H_{\text{coad}}$.

Then we have $B \simeq k^{\mathbb{A}_4}$, $k^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ or k^{D_n} , $n = 2, 3$ or 5 .

Lemma 6.11. *Suppose H is simple and $|G(H)| = 4$. Assume in addition that H has irreducible characters χ and χ' of degree 2, such that $G[\chi] \neq G[\chi']$. Then H is isomorphic to the self-dual Hopf algebra $\mathcal{B} = (kD_3 \otimes D_5)^J$.*

Proof. Since $G[\chi], G[\chi'] \neq 1$, then $G[\chi]$ and $G[\chi']$ are distinct subgroups of order 2. Then $B[\chi]$ and $B[\chi']$ are distinct Hopf subalgebras of H isomorphic to k^{D_3} or k^{D_5} .

Let $K[\chi] = k[G(H), B[\chi]]$ and $K[\chi'] = k[G(H), B[\chi']]$. Then $K[\chi], K[\chi']$ are Hopf subalgebras of H , whose irreducible comodules are of dimension 1 and 2. Hence $\dim K[\chi], \dim K[\chi']$ equal 12 or 20. Moreover, by dimension, $K[\chi]$ and $K[\chi']$ generate H as an algebra. In addition, $G(K[\chi]) = G(K[\chi']) = G(H)$ is not cyclic.

If one of them, say $K[\chi]$, is commutative, then $G[\chi']$ which is central in $K[\chi']$, would be central in H , whence H would not be simple. We may assume then that $K[\chi]$ and $K[\chi']$ are not commutative. Then there exists a twist $J \in kG(H) \otimes kG(H)$ such that $K[\chi]^J$ and $K[\chi']^J$ are group algebras [36, 5.2]. In particular, since $K[\chi]^J$ and $K[\chi']^J$ generate H^J as an algebra (because the algebra structure is unchanged under twisting), we find that H^J is cocommutative. The lemma follows from [17, Theorem 4.10]. \square

Corollary 6.12. *Suppose H is of type $(1, 4; 2, 6; 4, 2)$ as a coalgebra. If H is simple, then H is isomorphic to the self-dual Hopf algebra $\mathcal{B} = (kD_3 \otimes kD_5)^J$.*

Proof. Since the simple Hopf algebra \mathcal{B} is of type $(1, 4; 2, 6; 4, 2)$ as a coalgebra, in view of Lemma 6.11, it is enough to show that H has irreducible characters χ and χ' of degree 2, such that $G[\chi] \neq G[\chi']$.

In view of the coalgebra structure, there must exist irreducible characters χ and χ' of degree 2, such that $\chi\chi'$ is irreducible of degree 4. Otherwise, the sum of simple subcoalgebras of dimensions 1 and 2 would be a Hopf subalgebra of H of dimension 28, which is impossible. By [36, Theorem 2.4.2], we have $G[\chi] \cap G[\chi'] = 1$, thus $G[\chi]$, $G[\chi']$ are distinct subgroups of order 2. \square

7. APPENDIX: THE GROUP $\text{Opext}(k^N, k\mathbb{Z}_2)$

Suppose that $F = \mathbb{Z}_2$ is the cyclic group of order 2, and let N be a finite group. A matched pair (\mathbb{Z}_2, N) necessarily has trivial action \triangleright , whence the action $\triangleleft: N \times \mathbb{Z}_2 \rightarrow N$ is by group automorphisms.

Let $G = N \rtimes \mathbb{Z}_2$ be the corresponding semidirect product. Note that the restriction map $H^2(N \rtimes \mathbb{Z}_2, k^\times) \rightarrow H^2(\mathbb{Z}_2, k^\times) = 1$ is necessarily trivial.

By [49], there is an exact sequence

$$1 \rightarrow H^1(\mathbb{Z}_2, \widehat{N}) \rightarrow H^2(G, k^\times) \xrightarrow{\text{res}} H^2(N, k^\times)^{\mathbb{Z}_2} \xrightarrow{d_2} H^2(\mathbb{Z}_2, \widehat{N}) \rightarrow \widetilde{H}^3(G, k^\times).$$

Lemma 7.1. *Let (\mathbb{Z}_2, N) be a matched pair as above. Assume in addition that $H^2(N, k^\times) \simeq \mathbb{Z}_2$. Then we have a group isomorphism*

$$\text{Opext}(k^N, k\mathbb{Z}_2)/K \simeq 1, \text{ or } \mathbb{Z}_2,$$

where $K \simeq d_2(H^2(N, k^\times)) \subseteq H^2(\mathbb{Z}_2, \widehat{N})$.

The extensions corresponding to elements of K are, as Hopf algebras, twisting deformations of the split extension $k^N \# k\mathbb{Z}_2$.

Proof. To prove the lemma, we shall apply the results of [27], identifying $\text{Opext}(k^N, k\mathbb{Z}_2)$ with the group, denoted $H^2(k\mathbb{Z}_2, k^N)$ in *loc. cit.*, of classes of compatible cocycles (σ, τ) . Recall that there are subgroups $H_c^2(k\mathbb{Z}_2, k^N)$, $H_m^2(k\mathbb{Z}_2, k^N)$ of $\text{Opext}(k^N, k\mathbb{Z}_2)$, which are identified with the subgroup of classes of compatible pairs $(1, \tau)$ and $(\sigma, 1)$, respectively [27, Section 3].

In view of the assumption $H^2(N, k^\times) \simeq \mathbb{Z}_2$, we have $H^2(N, k^\times)^{\mathbb{Z}_2} = H^2(N, k^\times)$, and $H^1(\mathbb{Z}_2, H^2(N, k^\times)) \simeq \mathbb{Z}_2$.

Since k is algebraically closed, we have $\text{Opext}(k^N, k\mathbb{Z}_2) = H_c^2(k\mathbb{Z}_2, k^N)$, by [27, Theorem 4.4]. Then, by [27, Remark pp. 418],

$$H_m^2(k\mathbb{Z}_2, k^N) = H_m^2(k\mathbb{Z}_2, k^N) \cap H_c^2(k\mathbb{Z}_2, k^N) = d_2(H^2(N, k^\times)),$$

where $d_2 : H^2(N, k^\times) \rightarrow H^2(\mathbb{Z}_2, \widehat{N})$, is the connecting homomorphism in [49].

Let $K = H_m^2(k\mathbb{Z}_2, k^N)$. By [27, Theorem 7.1], we get an exact sequence

$$(7.1) \quad H^2(N, k^\times) \oplus K \xrightarrow{\Phi \oplus \iota} \text{Opext}(k^N, k\mathbb{Z}_2) \xrightarrow{\pi} \mathbb{Z}_2.$$

With the identification $\Phi = d_2$ as in [27], we see that

$$(\Phi \oplus \iota)(H^2(N, k^\times) \oplus K) = K \subseteq \text{Opext}(k^N, k\mathbb{Z}_2).$$

Thus, (7.1) induces an exact sequence

$$(7.2) \quad 1 \rightarrow K \rightarrow \text{Opext}(k^N, k\mathbb{Z}_2) \xrightarrow{\pi} \mathbb{Z}_2,$$

whence $\text{Opext}(k^N, k\mathbb{Z}_2)/K \simeq \pi(\text{Opext}(k^N, k\mathbb{Z}_2)) \leq \mathbb{Z}_2$.

According to [27, Remark pp. 418], the exact sequence (7.2) considered in the proof of Lemma 7.1 restricts, in this case, to the Kac exact sequence associated to the matched pair (\mathbb{Z}_2, N) [20], [29, Theorem 7.4]. In view of [30], all Hopf algebra extensions H arising from elements of K are, as Hopf algebras, twisting deformations (of the comultiplication) of the split extension. This finishes the proof of the lemma. \square

Remark 7.2. Keep the assumptions in Lemma 7.1. Since $H^2(N, k^\times) = \mathbb{Z}_2$, then either $K = 1$ or $K \simeq \mathbb{Z}_2$. The possibility $K = 1$ is equivalent to $d_2 = 1$, while $K \simeq \mathbb{Z}_2$ amounts to d_2 being injective.

Consider the restriction map $\text{res} : H^2(N \rtimes \mathbb{Z}_2, k^\times) \rightarrow H^2(N, k^\times)$. It follows from exactness of the sequence in [49], that $K = 1$ if and only if res is surjective, and $K \simeq \mathbb{Z}_2$ if and only if res is trivial.

In particular, if $H^2(\mathbb{Z}_2, \widehat{N}) = 1$ (for instance, if $|\widehat{N}|$ is odd), we have $\text{Opext}(k^N, k\mathbb{Z}_2) \simeq \pi(\text{Opext}(k^N, k\mathbb{Z}_2)) \leq \mathbb{Z}_2$.

Remark 7.3. Consider the matched pair (Γ, \mathbb{Z}_2) as in Section 5, where Γ is a nonabelian polyhedral group. The existence of the extensions $\mathcal{A}[\widetilde{\Gamma}]$, $\mathcal{B}[\widetilde{\Gamma}]$, shows that the map $\pi : \text{Opext}(k^\Gamma, k\mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ is in this case surjective. Therefore, we have an isomorphism $\text{Opext}(k^\Gamma, k\mathbb{Z}_2)/K \simeq \mathbb{Z}_2$.

REFERENCES

- [1] N. ANDRUSKIEWITSCH and J. DEVOTO, *Extensions of Hopf algebras*, St. Petersburg Math. J. **7** (1996), 17–52.
- [2] T. BANICA, *Le groupe quantique libre $U(n)$* , Comm. Math. Phys. **190** (1997), 143–172.
- [3] T. BANICA, *Symmetries of a generic coaction*, Math. Ann. **314** (1999), 763–780.
- [4] T. BANICA and J. BICHON, *Quantum groups acting on 4 points*, J. Reine Angew. Math. **626** (2009), 75–114.
- [5] J. BICHON, *Cosovereign Hopf algebras*, J. Pure Appl. Algebra **157** (2001), 121–133.
- [6] J. BICHON, *Cosemisimple Hopf algebras with antipode of arbitrary finite order*, New York J. Math. **8** (2002), 235–240.

- [7] J. BICHON, *The representation category of the quantum group of a non-degenerate bilinear form*, Commun. Algebra **31** (2003), 4831–4851.
- [8] J. BICHON, *Corepresentation theory of universal cosovereign Hopf algebras*, J. Lond. Math. Soc. **75** (2007), 83–98.
- [9] H. COXETER, *Regular complex polytopes*, Cambridge University Press, London (1974).
- [10] A. A. DAVYDOV, *Galois algebras and monoidal functors between categories of representations of finite groups*, J. Algebra **244** (2001), 273–301.
- [11] M. DIJKHUIZEN, *The double covering of the quantum group $SO_q(3)$* , Rend. Circ. Mat. Palermo (2) Suppl. **37** (1994), 47–57.
- [12] M. DUBOIS-VIOLETTE and G. LAUNER, *The quantum group of a non-degenerate bilinear form*, Phys. Lett. B **245**, 175–177 (1990).
- [13] P. ETINGOF and S. GELAKI, *The representation theory of cotriangular semisimple Hopf algebras*, Int. Math. Res. Not. **1999** (1999), 387–394.
- [14] P. ETINGOF, D. NIKSHYCH and V. OSTRIK, *On fusion categories*, Ann. Math. (2) **162** (2005), 581–642.
- [15] P. ETINGOF, D. NIKSHYCH and V. OSTRIK, *Weakly group-theoretical and solvable fusion categories*, preprint [arXiv:0809.3031v1](https://arxiv.org/abs/0809.3031v1) [math.QA].
- [16] L. FADDEEV, N. RESHETIKHIN and L. TAKHTADZHIAN, *Quantization of Lie groups and Lie algebras*, Leningr. Math. J. **1** (1990) 193–225.
- [17] C. GALINDO and S. NATALE, *Simple Hopf algebras and deformations of finite groups*, Math. Res. Lett. **14** (2007), 943–954.
- [18] S. GELAKI and D. NIKSHYCH, *Nilpotent fusion categories*, Adv. Math. **217** (2008), 1053–1071.
- [19] M. IZUMI and H. KOSAKI, *Kac algebras arising from composition of subfactors: general theory and classification*, Mem. Amer. Math. Soc. **158** (2002).
- [20] G. KAC, *Extensions of groups to ring groups*, Math. USSR. Sb. **5** (1968), 451–474.
- [21] Y. KASHINA, *Classification of semisimple Hopf algebras of dimension 16*, J. Algebra **232** (2000), 617–663.
- [22] Y. KASHINA, Y. SOMMERHÄUSER and Y. ZHU, *On higher Frobenius-Schur indicators*, Mem. Amer. Math. Soc. **855** (2006).
- [23] G. KARPILOVSKY, *Projective representations of finite groups*, Pure Appl. Math. **94**, Marcel Dekker, New York (1985).
- [24] A. KLIMYK and K. SCHMÜDGEN, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin (1997).
- [25] R. LARSON and D. RADFORD, *Semisimple cosemisimple Hopf algebras*, Amer. J. Math. **110** (1988), 187–195.
- [26] V. LINCENKO and S. MONTGOMERY, *A Frobenius-Schur theorem for Hopf algebras*, Algebr. Represent. Theory **3** (2000), 347–355.
- [27] M. MASTNAK, *Hopf algebra extensions arising from semi-direct products*, J. Algebra **251** (2002), 413–434.
- [28] A. MASUOKA, *Faithfully flat forms and cohomology of Hopf algebra extensions*, Comm. Algebra **25** (1997), 1169–1197.
- [29] A. MASUOKA, *Extensions of Hopf algebras*, Trab. Mat. **41/99**, FaMAF, 1999.
- [30] A. MASUOKA, *Hopf algebra extensions and cohomology*, in: New Directions in Hopf Algebras, MSRI Publ. **43** (2002), 167–209.
- [31] A. MASUOKA, *Cocycle deformations and Galois objects for some co-semisimple Hopf algebras of finite dimension*, Contemp. Math. **267** (2000), 195–214.
- [32] S. MONTGOMERY and S. WHITERSPOON, *Irreducible representations of crossed products*, J. Pure Appl. Algebra **129** (1998), 315–326.
- [33] M. MOVŠEV, *Twisting in group algebras of finite groups*, Func. Anal. Appl. **27** (1994), 240–244.
- [34] S. NATALE, *On semisimple Hopf algebras of dimension pq^2* , J. Algebra **221** (1999), 242–278.

- [35] S. NATALE, *Hopf algebras of dimension 12*, *Algebr. Repr, theory* **5** (2002), 445–455.
- [36] S. NATALE, *Semisolvability of semisimple Hopf algebras of low dimension*, *Mem. Amer. Math. Soc.* **186**, no. 874, (2007).
- [37] S. NATALE, *Hopf algebra extensions of group algebras and Tambara-Yamagami categories*, to appear in *Algebr. Represent. Theory*. Preprint [arXiv:0805.3172v1](https://arxiv.org/abs/0805.3172v1) (2008).
- [38] W. NICHOLS and M. RICHMOND, *The Grothendieck group of a Hopf algebra*, *J. Pure Appl. Algebra* **106** (1996), 297–306.
- [39] W. NICHOLS and M. ZOELLER, *A Hopf algebra freeness Theorem*, *Amer. J. Math.* **111** (1989), 381–385.
- [40] D. NIKSHYCH, *K_0 -rings and twisting of finite-dimensional semisimple Hopf algebras*, *Comm. Algebra* **26** (1998), 321–342.
- [41] J. PETIT, *Decomposition of the Turaev-Viro TQFT*, preprint [arXiv:0903.4512](https://arxiv.org/abs/0903.4512).
- [42] P. PODLEŚ, *Quantum spheres*, *Lett. Math. Phys.* **14** (1987), 193–202.
- [43] P. PODLEŚ, *Symmetries of Quantum Spaces. Subgroups and Quotient Spaces of Quantum SU(2) and SO(3) groups*, *Comm. Math. Phys.* **170** (1995), 1–20.
- [44] D. RADFORD, *On an analog of Lagrange’s theorem for commutative Hopf algebras*, *Proc. Amer. Math. Soc.* **79** (1980), 164–166.
- [45] H.-J. SCHNEIDER, *Some remarks on exact sequences of quantum groups*, *Comm. Algebra* **21** (1993), 3337–3357.
- [46] D. STEFAN, *Hopf algebras of low dimension*, *J. Algebra* **211** (1999), 343–361.
- [47] R. STEKOLSHCHIK, *Notes on Coxeter transformations and the McKay correspondence*, Springer Monographs in Mathematics, Berlin (2008).
- [48] M. SUZUKI, *Group Theory, I*, Springer-Verlag, Berlin (1982).
- [49] K. TAHARA, *On the second cohomology groups of semi-direct products*, *Math. Z.* **129** (1972), 365–379.
- [50] M. TAKEUCHI, *Quantum orthogonal and symplectic groups and their embedding into quantum GL*, *Proc. Japan Acad.* **65**, Ser. A (1989), 55–58.
- [51] M. TAKEUCHI, *q -representations of quantum groups*, *CMS Conf. Proc.* **16** (1995), 347–385.
- [52] S. WANG, *Free products of compact quantum groups*, *Comm. Math. Phys.* **167** (1995), 671–692.
- [53] W. WATERHOUSE, *The module structure of certain Hopf algebra extensions*, *Commun. Algebra* **10** (1982), 115–120.
- [54] S. ZHU, *On finite dimensional Hopf algebras*, *Comm. Algebra* **21** (1993), 3871–3885.

JULIEN BICHON: LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ BLAISE PASCAL,
 COMPLEXE UNIVERSITAIRE DES CÉZEAUX, 63177 AUBIÈRE CEDEX, FRANCE
E-mail address: Julien.Bichon@math.univ-bpclermont.fr
URL: <http://www.math.univ-bpclermont.fr/~bichon>

SONIA NATALE: FACULTAD DE MATEMÁTICA, ASTRONOMÍA Y FÍSICA. UNIVERSIDAD NACIONAL DE CÓRDOBA. CIEM – CONICET. (5000) CIUDAD UNIVERSITARIA. CÓRDOBA, ARGENTINA
E-mail address: natale@famaf.unc.edu.ar
URL: <http://www.famaf.unc.edu.ar/~natale>