Moment systems derived from relativistic kinetic equations

Véronique Bagland, Pierre Degond and Mohammed Lemou

Mathématiques pour l'Industrie et la Physique, CNRS UMR 5640,
Université Paul Sabatier – Toulouse 3,
118 route de Narbonne, F–31062 Toulouse cedex 4, France
E-mail: bagland@mip.ups-tlse.fr, degond@mip.ups-tlse.fr, lemou@mip.ups-tlse.fr

Abstract

In this paper, we are interested in the derivation of macroscopic equations from kinetic ones using a moment method in a relativistic framework. More precisely, we establish the general form of moments that are compatible with the *Lorentz invariance* and derive a hierarchy of relativistic moment systems from a Boltzmann kinetic equation. The proof is based on the representation theory of Lie algebras. We then extend this derivation to the classical case and general families of moments that obey the *Galilean invariance* are also constructed. It is remarkable that the set of formal classical limits of the so-obtained relativistic moment systems is not identical to the set of classical moments quoted in [17] and one could use a new physically relevant criterion to derive suitable moment systems in the classical case. Finally, the ultra-relativistic limit is considered.

1 Introduction

Particle systems may be modelled at many different levels (microscopic, mesoscopic or macroscopic) depending on the scale of the studied physical phenomena and on the desired degree of accuracy for its description. In many situations, the precise knowledge of some physical quantities (density, momentum, energy, viscosity, heat flux, etc) is crucial and one cannot use the standard Euler or Navier-Stokes equations to describe such quantities. On the other hand the use of more refined models as kinetic equations is too expensive in general and makes extremely slow any realistic and accurate numerical simulation. This is due to the complexity of the kinetic equation (coupling Vlasov equation with Poisson or Maxwell equations) and to the number of involved variables (one time variable plus six space-velocity coordinates). Therefore, it is necessary in general to derive more reduced models from kinetic equations which are able to describe the desired physical quantities with a sufficient degree of accuracy. This has been a challenging subject of a large number of works in the past and still stimulates many current researches.

There are mainly two approaches to derive macroscopic equations from kinetic ones. The first one consists in deriving Euler or Navier-Stokes like equations with various expressions for the viscosity and the heat flux. This strategy supposes that the particle distribution function is close to the so-called thermodynamical equilibrium and can be expanded into successive approximations about this equilibrium according to the well known Chapman-Enskog or Hilbert procedures. The second strategy consists in directly deriving systems of equations involving the desired macroscopic quantities (mass, momentum, energy, etc), which are moments of the distribution function with respect to the velocity variable. To close the obtained systems, this strategy also needs an assumption on the distribution function which is not necessarily close to the equilibrium. For instance, Grad [8] uses an expansion in terms of Hermite polynomials whereas in [17, 18], the closure is based on the entropy minimization principle. In this last strategy, a first and important step is to derive suitable sets of moments in the velocity space, that is sets which are compatible with the *Galilean invariance* in the classical case and with the *Lorentz invariance* in the relativistic case. To our knowledge, the case of the Lorentz invariance has not been considered yet. Our work is intended to investigate this problem.

In this paper, the general form of moment spaces that satisfy the Lorentz invariance is determined. More precisely, we give the more general form of finite dimensional spaces of polynomial functions of the energy and momentum which obey *the Lorentz* invariance. The proof is based on the representation theory of Lie algebras [6, 10]. We then consider the classical and ultra-relativistic limits of these spaces. For the sake of completeness, we also give a similar result for Galilean invariant spaces in the appendix.

Hierarchies of moment systems have already been derived in several works in both the relativistic and the classical cases, and we refer the reader to [2, 4, 9, 13, 18] and references therein for detailed descriptions. In these works, various closure strategies are used and the question of classical and ultra-relativistic limits is also investigated. However, the problem of deriving a general form of Lorentz invariant sets of moments has not been addressed at a rigorous level. Our purpose here is to give a rigorous basis and a systematic way to select the families of moments that are compatible with the Lorentz invariance principle. Classical and ultra-relativistic limits of the so obtained systems are also discussed.

Before going to the presentation of our main results and for the sake of self consistency, we first recall some basic notions in relativistic mechanics. For much more detailed and complete presentations, we refer to [9, 14].

1.1 The relativistic kinetic model

Unlike classical mechanics where time is absolute, that is independent of the frame, a time is attached to each frame in relativistic mechanics. Therefore, the position of a particle is defined by its temporal and spatial coordinates. Let $\mathcal{R}$ and $\mathcal{R}'$ be two inertial frames such that $\mathcal{R}'$ moves with the velocity u with respect to $\mathcal{R}$. Denote by (t, x) and (t', x') the time-space coordinates respectively in $\mathcal{R}$ and $\mathcal{R}'$. Then, the change of frames is given by

$$t = \gamma_u \left(t' + \frac{u \cdot x'}{c^2} \right) \qquad \text{and} \qquad x = x' + (\gamma_u - 1)\,(u \cdot x')\, \frac{u}{|u|^2} + \gamma_u\, u\, t', \tag{1}$$

where c denotes the speed of light and

$$\gamma_u = \left(1 - \frac{|u|^2}{c^2} \right)^{-1/2}.$$

The vector $\vec{x} = (ct, x)$ is called the radius four-vector in $\mathcal{R}$. Let $\vec{x}\,' = (ct', x')$ denote the radius four-vector in $\mathcal{R}'$. Then, (1) reads $\vec{x} = L_u \vec{x}\,'$, with

$$L_u \vec{a} = \left(\gamma_u \left(a^0 + \frac{u \cdot a}{c} \right), a + (\gamma_u - 1)\,(u \cdot a)\, \frac{u}{|u|^2} + \gamma_u\, \frac{u}{c}\, a^0 \right), \tag{2}$$

where $\vec{a} = (a^j)_{0 \le j \le 3} = (a^0, a)$ with $a = (a^j)_{1 \le j \le 3}$. The function L_u is called the proper Lorentz transformation associated to the velocity u. By analogy, any vector $\vec{y} = (y^j)_{0 \le j \le 3}$ whose components transform like those of $\vec{x}$ under a change of inertial frame is called a four-vector. An important four-vector is the energy-momentum four-vector $\vec{p} = (\varepsilon/c,\, p)$, where

$$\varepsilon = \gamma m c^2 \qquad \text{and} \qquad p = \gamma m v, \qquad \text{with} \qquad \gamma = \left(1 - \frac{|v|^2}{c^2} \right)^{-\frac{1}{2}}, \tag{3}$$

denote respectively the energy and the momentum of a relativistic particle with mass m and velocity v. It also reads

$$\varepsilon = c\sqrt{m^2 c^2 + |p|^2} \qquad \text{and} \qquad v = \frac{p}{m\sqrt{1 + \frac{|p|^2}{m^2 c^2}}}. \tag{4}$$

Similarly, any tensor of rank n whose components transform like those of the tensor product of n four-vectors under a change of inertial frame is called a four-tensor.

Let us now recall that kinetic theory generalizes to the relativistic case (see [9, 14]). During an elastic collision between two relativistic particles with momenta p and p_*, the conservation of momentum and energy holds, that is

$$p + p_* = p^\natural + p_*^\natural \qquad \text{and} \qquad \varepsilon(p) + \varepsilon(p_*) = \varepsilon(p^\natural) + \varepsilon(p_*^\natural), \tag{5}$$

where $p^\natural$ and $p_*^\natural$ denote the post-collisional momenta. As in the classical case, we may then derive the relativistic Boltzmann equation, which reads (see [7, 9, 14])

$$\partial_t f + v \cdot \nabla_x f = Q_R(f,f), \tag{6}$$

with

$$Q_R(f,f) = \iint_{\mathbb{S}^2 \times \mathbb{R}^3} \sigma(p, p_*, p^\natural, p_*^\natural)\, v_M(p,p_*)\, (f^\natural f_*^\natural - f f_*)\, dp_*\, d\omega, \tag{7}$$

where $f = f(t,x,p)$, $f_* = f(t,x,p_*)$, $f^\natural = f(t,x,p^\natural)$, $f_*^\natural = f(t,x,p_*^\natural)$, σ denotes the cross-section, v_M the Møller velocity,

$$v_M(p,p_*) = |v_{rel}|\, \frac{\varepsilon\varepsilon_* - c^2\, p\cdot p_*}{\varepsilon\varepsilon_*} = \left(|v - v_*|^2 - \frac{|v\times v_*|^2}{c^2}\right)^{1/2}, \tag{8}$$

and $d\omega$ is an element of solid angle in the centre of mass system. The structure of the relativistic Boltzmann equation (6) is similar to the classical one. Its relativistic nature appears in the relationship (4) between momentum and velocity and in the definition of the Møller velocity (8). This relativistic aspect also appears implicitly in the definition of σ, which is a non-negative function of the energy s and the deviation angle θ (in the centre of mass system), both given by

$$s = \frac{(\varepsilon+\varepsilon_*)^2}{c^2} - |p + p_*|^2,$$

and

$$\cos\theta = \frac{\left(\varepsilon - \varepsilon_*\right)\left(\varepsilon^\natural - \varepsilon_*^\natural\right) - c^2\left(p - p_*\right)\cdot\left(p^\natural - p_*^\natural\right)}{\left(\varepsilon-\varepsilon_*\right)^2 - c^2\,|p - p_*|^2}.$$

In the case of charged particles (also called the Coulomb case), the cross-section σ reads, in the centre of mass system, (see [1, Section 81, Problem 6]),

$$\sigma = \left(\frac{qq_*}{4\pi\varepsilon_0}\right)^2 \frac{1}{8c^4(\overline{\varepsilon}+\overline{\varepsilon}_*)^2|\overline{p}|^4\sin^4(\theta/2)} \times \left((\overline{\varepsilon}\,\overline{\varepsilon}_* + c^2|\overline{p}|^2)^2 + (\overline{\varepsilon}\,\overline{\varepsilon}_* + c^2|\overline{p}|^2\cos\theta)^2 - 2(m^2+m_*^2)c^6|\overline{p}|^2\sin^2(\theta/2)\right), \tag{9}$$

where $(\overline{\varepsilon}/c, \overline{p})$ and $(\overline{\varepsilon}_*/c, \overline{p}_*)$ denote respectively the energy-momentum four-vectors $\vec{p}$ and $\vec{p}_*$ in the centre of mass system (in this system, we have $\overline{p} = -\overline{p}_*$ and then $|\overline{p}| = |\overline{p}_*|$).

Let us point out that, as for the classical Boltzmann equation, the mass, momentum and energy are locally conserved quantities for (6)-(7) and that the relativistic Boltzmann equation possesses an entropy. In the relativistic case, the jacobian of the application $(p,p_*) \mapsto (p^\natural, p_*^\natural)$ is not equal to 1. However, since $v_M(p,p_*) = v_M(p^\natural, p_*^\natural)\, \frac{\partial(p^\natural,p_*^\natural)}{\partial(p,p_*)}$, we still have the following weak formulation,

$$\int_{\mathbb{R}^3} Q_R(f,f)\,\varphi\, dp = \frac{1}{4}\iiint_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} \sigma\, v_M\, (f^\natural f_*^\natural - f f_*)(\varphi + \varphi_* - \varphi^\natural - \varphi_*^\natural)\, dp\, dp_*\, d\omega.$$

We then infer from (5) that 1, p and ε are locally conserved quantities. Moreover, choosing $\varphi = \ln f$, we obtain the local dissipation law of the entropy $S(f) = \int_{\mathbb{R}^3} (f \ln f - f)\, dp$, that is

$$\partial_t S(f) + \nabla_x \cdot \int_{\mathbb{R}^3} v\,(f \ln f - f)\, dp = \int_{\mathbb{R}^3} Q_R(f,f) \ln f \, dp \le 0. \tag{10}$$

Equilibrium states of (6) are defined to be the functions that cancel the right hand side of (10) or, equivalently, the functions $f \ge 0$ such that $Q_R(f,f) = 0$. They are the local relativistic Maxwellians

$$\mathcal{M}(p) = A \exp\big(- \beta^0\, \varepsilon(p) + \beta \cdot p \big) \qquad \text{with} \qquad A \in \mathbb{R}_+, \beta^0 \in \mathbb{R}_+, \, \beta \in \mathbb{R}^3. \tag{11}$$

1.2 Setting of the problem

Formally, multiplying (6) by 1, p and ε, integrating with respect to p and closing this system with the Maxwellian (11) that minimizes the entropy at fixed mass, momentum and energy, we recover the relativistic hydrodynamic equations. We are looking here for moment spaces $\mathbb{M}$ that generalize the fluid dynamic approximation and thus that contain 1, p and ε.

Moreover, the space $\mathbb{M}$ ought to respect physical symmetries. A specificity of the relativistic case is that the Galilean invariance is replaced by the Lorentzian invariance. More precisely, let L be either a proper Lorentz transformation or a rotation of the axis of the spatial coordinate system, that is L is either defined by (2) for some $u \in \mathbb{R}^3$, $|u| < c$ or given by

$$L = \begin{pmatrix} 1 & 0 \\ 0 & O \end{pmatrix}, \tag{12}$$

where O is a 3-dimensional orthogonal matrix. Then, L corresponds either to a change of frame or to a change of axis in the momentum space. Let us denote respectively by $\vec{x}\,'$ and $\vec{p}\,'$ the radius and the energy-momentum four-vectors in the new system of coordinates. We have $\vec{x} = L^{-1}\vec{x}\,'$, $\vec{p} = L^{-1}\vec{p}\,'$,

$$\frac{dp}{\gamma(p)} = \frac{dp'}{\gamma(p')}, \qquad \sigma(p, p_*, p^\natural, p_*^\natural) = \sigma(p', p'_*, p'^\natural, p'^\natural_*), \tag{13}$$

and

$$\begin{aligned} \gamma(p) v_M(p,p_*) dp_* &= |v_{rel}|(\varepsilon\varepsilon_* - c^2\, p \cdot p_*)\, \frac{dp_*}{m^2 c^4 \gamma(p_*)} \\ &= |v_{rel}|(\varepsilon'\varepsilon'_* - c^2\, p' \cdot p'_*)\, \frac{dp'_*}{m^2 c^4 \gamma(p'_*)} = \gamma(p') v_M(p', p'_*) dp'_*, \end{aligned} \tag{14}$$

where

$$\gamma(p) = \sqrt{1 + \frac{|p|^2}{m^2 c^2}}.$$

Therefore, if f denotes a solution to (6)-(7) then the function f' defined in the new system of coordinates by $f'(t', x', p') = f(t, x, p)$ is a solution to

$$\partial_{t'} f' + v(p') \cdot \nabla_{x'} f' = Q_R(f', f').$$

This corresponds to the Lorentzian invariance. The translations and the rotations that we consider in the classical case are replaced, in the relativistic case, by the proper Lorentz transformations and the rotations of the form (12). We want the space $\mathbb{M}$ to be compatible with this invariance. More precisely, let $(\varphi_1(\vec{p}), \ldots, \varphi_N(\vec{p}))$ be a moment basis for $\mathbb{M}$. Using the radius four-vector $\vec{x} = (x^j)_{0 \le j \le 3}$ and the energy-momentum four-vector $\vec{p} = (p^j)_{0 \le j \le 3}$, the Boltzmann equation (6) also reads

$$p^j \frac{\partial f}{\partial x^j} = m\, \gamma(p)\, Q_R(f, f). \tag{15}$$

Here as in the rest of this paper, we make use of the Einstein summation convention. Multiplying (15) by $\varphi_k(\vec{p})/\gamma(p)$ for some $k \in [\![1, N]\!]$ and integrating with respect to p, we obtain

$$\frac{\partial}{\partial x^j} \int_{\mathbb{R}^3} p^j \varphi_k(\vec{p}) f(\vec{x}, \vec{p}) \frac{dp}{\gamma(p)} = m \int_{\mathbb{R}^3} \varphi_k(\vec{p}) Q_R(f, f)(\vec{x}, \vec{p}) dp, \qquad k \in [\![1, N]\!]. \tag{16}$$

We set $\tilde{\varphi}_k = \varphi_k \circ L^{-1}$. Then, we deduce from (13) and (14) that, in the new system of coordinates, (16) reads

$$\frac{\partial}{\partial x'^j} \int_{\mathbb{R}^3} p'^j \tilde{\varphi}_k(\vec{p}\,') f'(\vec{x}\,', \vec{p}\,') \frac{dp'}{\gamma(p')} = m \int_{\mathbb{R}^3} \tilde{\varphi}_k(\vec{p}\,') Q_R(f', f')(\vec{x}\,', \vec{p}\,') dp', \qquad k \in [\![1, N]\!].$$

Consequently, a moment space $\mathbb{M}$ is said to be compatible with the Lorentzian invariance if there exist some constants $\lambda_{j,k}$ such that $\tilde{\varphi}_k = \sum_{j=1}^N \lambda_{j,k} \varphi_j$ for $k = 1, \ldots, N$. We are looking here for spaces that are invariant under any proper Lorentz transformation and any rotation in the momentum space.

Moreover, as in [17], we introduce the convex cone

$$\mathbb{M}_c := \left\{ r \in \mathbb{M} : \int_{\mathbb{R}^3} \exp(r(\varepsilon(p), p))\, dp < \infty \right\},$$

for every space $\mathbb{M}$ constituted of functions of p and ε. In [17], Levermore introduced admissible moment spaces. A moment space $\mathbb{M}$ is said to be admissible if the associated cone $\mathbb{M}_c$ has a nonempty interior in $\mathbb{M}$. We are only interested in admissible spaces.

Summarizing, we are looking for finite dimensional spaces $\mathbb{M}$ of polynomial functions of the energy and momentum that satisfy

(I) $\mathrm{span}\,\big(1,\, p,\, \varepsilon\big) \subset \mathbb{M}$,

(II) $\mathbb{M}$ is invariant under any proper Lorentz transformation and any rotation in the momentum space,

(III) the cone $\mathbb{M}_c$ has a nonempty interior in $\mathbb{M}$.

Here as in the rest of the paper, the span notation is applied to a collection of scalars, vectors and tensors and means all linear combinations of their components.

In order to construct spaces satisfying conditions (I), (II) and (III), a first idea is to consider tensor products of the four-vector $\vec{p}$. Thus, for every $n \in \mathbb{N}_*$, we set

$$\mathcal{T}_n(\vec{p}) = \otimes^n \vec{p}, \tag{17}$$

and denote by $\mathbb{P}_n$ the space generated by the components of $\mathcal{T}_n$. We point out that each $\mathbb{P}_n$ satisfies condition (II). Since span$(1,\, p,\, \varepsilon)$ is itself invariant under any proper Lorentz transformation and any rotation in the momentum space, we set, for every $n \in \mathbb{N}_*$,

$$\tilde{\mathbb{P}}_n = \operatorname{span}(1, p, \varepsilon, \mathcal{T}_n). \tag{18}$$

It only remains to check that condition (III) holds. Given $r \in \tilde{\mathbb{P}}_n$,

$$r = -\sum_{(i_1,\dots,i_n)\in[\![0,3]\!]^n} \alpha_{i_1,\dots,i_n} \mathcal{T}_n^{i_1,\dots,i_n}(\vec{p}) + \beta + \gamma\cdot p + \delta\varepsilon,$$

it suffices to suppose that the coefficient $\alpha_{0,\dots,0}$ in front of $\mathcal{T}_n^{0,\dots,0}(\vec{p}) = (\varepsilon/c)^n$ is large enough so that r belongs to $\tilde{\mathbb{P}}_{n_c}$. Consequently, the space $\tilde{\mathbb{P}}_n$ fulfils each of our requirements. Moreover, we point out that any vector sum of the spaces $\tilde{\mathbb{P}}_n$ also satisfies conditions (I), (II) and (III).

We notice that, contrary to the classical case where tensor products of the velocity vector are independent (up to symmetries), tensor products of the energy-momentum four-vector are not independent. Indeed, any component of $\mathcal{T}_{n-2k}$ may be written as a linear combination of components of $\mathcal{T}_n$, for every $k \in [\![0, [n/2]]\!]$, where $[x]$ denotes the integer part of x. Indeed, we have

$$\mathcal{T}_{n-2}^{i_1,\dots,i_{n-2}} = \frac{1}{m^2\,c^2} \sum_{(i,j)\in[\![0,3]\!]^2} g_{i,j}\, \mathcal{T}_n^{i_1,\dots,i_{n-2},i,j},$$

where $(i_1,\dots,i_{n-2}) \in [\![0,3]\!]^{n-2}$, $g_{0,0} = 1$, $g_{1,1} = g_{2,2} = g_{3,3} = -1$ and $g_{i,j} = 0$ for $i \neq j$. We say that $m^2c^2\mathcal{T}_{n-2}$ is obtained from $\mathcal{T}_n$ by contraction with respect to the last two indices. Thus, any component of $\mathcal{T}_{n-2k}$, for $k \in [\![0, [n/2]]\!]$, belongs to $\tilde{\mathbb{P}}_n$. By contrast to the classical case, we do not obtain any additional moment spaces with the contraction operation.

However, a second idea to construct spaces that satisfy condition (II) is by considering the orthogonal complement of $\mathbb{P}_{n-2}$ into $\mathbb{P}_n$. More precisely, let us consider the tensor space, spanned by $\mathcal{T}_n$ and all the tensors obtained from $\mathcal{T}_n$ by contraction over an arbitrary number of pairs of indices (plus symmetrization). Let us denote this tensor space by $\mathbb{T}_n$. Obviously, from the remark above, the components of the tensors in $\mathbb{T}_n$ span the polynomial space $\mathbb{P}_n$, as much as the components of the sole tensor $\mathcal{T}_n$ did. However, within $\mathbb{T}_n$, we can consider the subspace $\hat{\mathbb{T}}_n$ of tensors spanned by all contractions of $\mathcal{T}_n$ over **at least** one pair of indices (plus symmetrization). Obviously, again using the remark above, the components of $\hat{\mathbb{T}}_n$ span the polynomial space $\mathbb{P}_{n-2}$. The orthogonal supplement of $\hat{\mathbb{T}}_n$ in $\mathbb{T}_n$ (with respect to the Minkowski inner product on tensors induced by the Minkowski inner product on four-vectors) is a one dimensional vector space spanned by a symmetric tensor $S_n(\vec{p})$. The components of $S_n(\vec{p})$ form a Lorentz invariant space denoted by $\mathbb{M}_n$ which is strictly included in $\mathbb{P}_n$ as soon as $n \geq 2$. We shall actually prove that $\mathbb{M}_n$ is an irreducible Lorentz invariant space, i.e. there is no non-trivial invariant subspace strictly included in $\mathbb{M}_n$.

Now, setting $\tilde{\mathbb{M}}_n = \mathbb{M}_n + \text{span}(1, p, \varepsilon)$, we construct a moment space which satisfies criteria (I) and (II). It remains to check that it satisfies condition (III). In fact, condition (III) is satisfied as much as for $\tilde{\mathbb{P}}_n$ since $\tilde{\mathbb{M}}_n$ differs from $\tilde{\mathbb{P}}_n$ by combinations of monomials of (ε, p) of degree $n-2$ at most. Therefore, $\tilde{\mathbb{M}}_n$ is an admissible moment space with no non-trivial admissible moment space strictly included in it.

To show these results, we first use the representation theory of Lie groups and Lie algebras to determine all irreducible spaces that satisfy condition (II) (cf. Theorem 1 and Proposition 2). Then, in Theorem 3 we identify these spaces with those obtained through the construction detailed above. In Section 3, we consider the classical limit of these relativistic moment spaces and suggest a new criterion for choosing moment spaces in the classical case. In Section 4, we are interested in the ultra-relativistic case. We then present the moment closure problem in Section 5. For the sake of completeness, the proof of Theorem 1 is given in Section 6. Finally, the representation theory of Lie groups and Lie algebras may also be used, in the classical case, to determine the spaces that are invariant under any rotation and this is stated in the appendix.

2 Moment system hierarchy and Lorentz invariance

We are looking for the finite dimensional subspaces of $\mathbb{R}[\varepsilon, p^1, p^2, p^3]$ that satisfy conditions (I), (II) and (III). We consider the Minkowski space $\mathbb{R}^4$ endowed with the non-degenerate symmetric bilinear form g defined by

$$g(a, b) = a^0\, b^0 - a^1\, b^1 - a^2\, b^2 - a^3\, b^3, \qquad a,\, b \in \mathbb{R}^4.$$

The set of real matrices $L \in \mathcal{M}(4, \mathbb{R})$ that leave g invariant (i.e. such that $g(Lx, Ly) = g(x, y)$ for all $x, y \in \mathbb{R}^4$) forms the generalized orthogonal group $O(1, 3)$. The set of matrices L from $O(1, 3)$ such that $\det(L) = 1$ and $L_{00} \geq 1$ (i.e. there is no time inversion) is called the proper Lorentz group and denoted by $SO(1, 3)_e$. This group is generated by the proper Lorentz transformations and the rotations in the momentum space. Therefore, we consider the following action of $SO(1,3)_e$ on the subspace $\mathcal{P}_n$ composed of the polynomials of $\mathbb{R}[y_0, y_1, y_2, y_3]$ with total degree less or equal to n,

$$\begin{aligned} \varphi : SO(1,3)_e &\longrightarrow GL(\mathcal{P}_n) \\ L &\longmapsto \{R(y_0, y_1, y_2, y_3) \mapsto R(L^{-1}(y_0, y_1, y_2, y_3))\}. \end{aligned} \tag{19}$$

Finding the finite dimensional subspaces of $\mathbb{R}[\varepsilon, p^1, p^2, p^3]$ that satisfy condition (II) amounts to finding the irreducible subrepresentations of $(\varphi, \mathcal{P}_n)$. This is the aim of the following theorem, which rests on the representation theory of Lie groups and Lie algebras. Its proof is postponed to Section 6.

Theorem 1 *A space W is an irreducible subrepresentation of $(\varphi, \mathcal{P}_n)$ if and only if there exist $j \in [\![0, [n/2]]\!]$ and some real numbers $(\lambda_k)_{0 \le k \le j}$ such that W is generated by the real parts and the imaginary parts of*

$$\sum_{k=0}^{j} \lambda_k (y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k} \sum_{m=\max(q-r,0)}^{q} \frac{(n-2j-r)!\, r!}{(n-2j-r-m)!(r-q+m)!} \binom{q}{m} \\ (y_0 + y_3)^m (y_0 - y_3)^{r-q+m} (y_1 + iy_2)^{n-2j-r-m} (y_1 - iy_2)^{q-m}, \tag{20}$$

for $q, r \in [\![0, n-2j]\!]$, $q + r \le n - 2j$. Moreover, any subrepresentation of $(\varphi, \mathcal{P}_n)$ is a direct sum of irreducible subrepresentations.

Here as in the rest of this paper, $[x]$ denotes the integer part of $x \in \mathbb{R}$ and $\binom{q}{m}$ stands for the binomial coefficient. This theorem describes all the irreducible representations of $(\varphi, \mathcal{P}_n)$. We deduce then all the spaces that satisfy condition (II) by replacing (y_0, y_1, y_2, y_3) with $(\varepsilon/c, p^1, p^2, p^3)$. We notice that, since $(\varepsilon/c)^2 - |p|^2 = m^2c^2$, the factor $(y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k}$ in (20) is replaced with the constant $(m^2c^2)^{j-k}$. Therefore, we have the following proposition.

Proposition 2 *For every $l \in \mathbb{N}$, let $\mathbb{M}_l$ denote the vector space generated by the real parts and the imaginary parts of*

$$\sum_{m=\max(q-r,0)}^{q} \frac{(l-r)!\, r!}{(l-r-m)!(r-q+m)!} \binom{q}{m} \\ (\varepsilon/c + p^3)^m (\varepsilon/c - p^3)^{r-q+m} (p^1 + ip^2)^{l-r-m} (p^1 - ip^2)^{q-m}, \tag{21}$$

for $q, r \in [\![0, l]\!]$, $q + r \le l$. Each $\mathbb{M}_l$ satisfies condition (II) and is irreducible, i.e. there is no non-trivial strict subspace of $\mathbb{M}_l$ which is invariant under the proper Lorentz group. Moreover, a finite dimensional subspace $\mathbb{M}$ of $\mathbb{R}[\varepsilon, p^1, p^2, p^3]$ satisfies condition (II) if and only if there exist $N \in \mathbb{N}$ and some $l_k \in \mathbb{N}$, $k = 1, \dots, N$ such that $\mathbb{M}$ is the direct sum of the $\mathbb{M}_{l_k}$, $k = 1, \dots, N$.

Proof. It is clear that $\mathbb{M}_l$ satisfies condition (II). Let us assume that $\mathbb{M}_l$ is not irreducible. Then, there exists a non-trivial subspace Q of $\mathbb{M}_l$ that is stable under the proper Lorentz group. Let R be a supplement of Q in $\mathbb{M}_l$. Then, $\mathbb{M}_l = Q \oplus R$. Let $(q_s)_{1\le s\le S}$ be a basis of Q and $(r_t)_{1\le t\le T}$ a basis of R. Let π denote the function defined by $\pi(P(y_0,y_1,y_2,y_3)) = P(\varepsilon/c,p^1,p^2,p^3)$, for every $P \in \mathcal{P}_l$. Then, $\mathbb{M}_l = \pi(\Gamma_l)$, where Γ_l is the real vector space generated by the real parts and the imaginary parts of (20) with $n = l$ and $j = 0$. For every $1 \le s \le S$ and $1 \le t \le T$, there exist $\tilde{q}_s, \tilde{r}_t \in \Gamma_l$ such that $q_s = \pi(\tilde{q}_s)$ and $r_t = \pi(\tilde{r}_t)$. Let us denote by $\tilde{Q}$ and $\tilde{R}$ the vector spaces generated respectively by $(\tilde{q}_s)_{1\le s\le S}$ and $(\tilde{r}_t)_{1\le t\le T}$. We have $\ker(\pi) = (y_0^2 - y_1^2 - y_2^2 - y_3^2 - m^2c^2)\mathcal{P}_{l-2}$ and we deduce from (57) that

$$\mathcal{P}_l = \oplus_{k=0}^{l}\Gamma_k \oplus \ker(\pi), \tag{22}$$

because $\Gamma_k = \tilde{\Gamma}^{(k)}_{k,0}$ for $k \in [\![0,l]\!]$. Thus, $\pi_{|\Gamma_l}$ is an injection and $\dim(\Gamma_l) = \dim(\mathbb{M}_l)$. Consequently, $\Gamma_l = \tilde{Q} \oplus \tilde{R}$ and $\tilde{Q}$ is a non-trivial subspace of Γ_l that is stable under the proper Lorentz group, which contradicts the irreducibility of Γ_l.

Let $\mathbb{M}$ be a finite dimensional subspace of $\mathbb{R}[\varepsilon/c,p^1,p^2,p^3]$ that satisfies condition (II). It remains to check that $\mathbb{M}$ is a direct sum of spaces $\mathbb{M}_l$. We denote by $(m_i)_{1\le i\le N}$ a basis of $\mathbb{M}$. For every $i \in [\![1,N]\!]$, let $\tilde{m}_i \in \mathbb{R}[y_0,y_1,y_2,y_3]$ such that $\pi(\tilde{m}_i) = m_i$. Let $\tilde{\mathbb{M}} \subset \mathbb{R}[y_0,y_1,y_2,y_3]$ be the smallest vector space that contains all the $\tilde{m}_i$ and is stable under the proper Lorentz group. We have $\tilde{\mathbb{M}} \subset \mathcal{P}_n$ for some $n \in \mathbb{N}$. By Theorem 1, $\tilde{\mathbb{M}} = \oplus_\alpha \Gamma_\alpha$, where $(\Gamma_a)_\alpha$ denote irreducible subrepresentations of $(\varphi,\mathcal{P}_n)$. The vector space $\pi(\tilde{\mathbb{M}})$ contains $\mathbb{M}$. Let $m \in \pi(\tilde{\mathbb{M}})$. Then, $m = \pi(\tilde{m})$ with $\tilde{m} \in \tilde{\mathbb{M}}$. Thus,

$$\tilde{m}(y_0,y_1,y_2,y_3) = \sum_{i=1}^{N} \sum_{L\in SO(1,3)_e} \lambda_{i,L}\, \tilde{m}_i(L^{-1}(y_0,y_1,y_2,y_3)),$$

where the coefficients $\lambda_{i,L}$ are almost all null. Consequently,

$$m(\varepsilon/c,p^1,p^2,p^3) = \sum_{i=1}^{N} \sum_{L\in SO(1,3)_e} \lambda_{i,L}\, m_i(L^{-1}(\varepsilon/c,p^1,p^2,p^3)).$$

Hence $m \in \mathbb{M}$ and $\pi(\tilde{\mathbb{M}}) = \mathbb{M}$. We thus deduce that $\mathbb{M} = \sum_\alpha \pi(\Gamma_\alpha) = \sum_{k=1}^{N} \mathbb{M}_{l_k}$, where we have only kept the spaces $\mathbb{M}_\alpha$ that are distinct. By (22), the spaces $(\mathbb{M}_k)_{0\le k\le n}$ are linearly independent, which completes the proof of Proposition 2. □

We now make the connection between $\mathbb{M}_l$ and the orthogonal complement construction stated at the end of Section 1.2. Before stating the result, we introduce some notations. For any tensor T of order k, we denote by $\overline{T}$ the symmetric part of T, that is the tensor whose components are

$$\overline{T}^{j_1,\dots,j_k} = \frac{1}{k!}\sum_{\sigma\in\Sigma_k} T^{j_{\sigma(1)},\dots,j_{\sigma(k)}}, \qquad (j_1,\dots,j_k) \in [\![0,3]\!]^k,$$

where Σ_k denotes the symmetric group of order k. Let $l \in \mathbb{N}$. We recall that $\mathcal{T}_l$ is defined by (17). We define

$$\begin{aligned} \mathbb{T}_l &= \operatorname{span}\Big(\overline{\underbrace{g\otimes\ldots\otimes g}_{k \text{ times}}\otimes\mathcal{T}_{l-2k}(\vec{p})}, 0\le k\le [l/2]\Big), \\ \hat{\mathbb{T}}_l &= \operatorname{span}\Big(\overline{\underbrace{g\otimes\ldots\otimes g}_{k \text{ times}}\otimes\mathcal{T}_{l-2k}(\vec{p})}, 1\le k\le [l/2]\Big). \end{aligned}$$

Obviously $\mathbb{T}_l = \hat{\mathbb{T}}_l \oplus \mathcal{T}_l\,\mathbb{R}$. Finally, the Minkowski form on tensors induced by the Minkowski form on four-vectors is given by

$$\langle T_1, T_2\rangle = g_{i_1,j_1}\ldots g_{i_l,j_l}\, T_1^{i_1,\ldots,i_l}\, T_2^{j_1,\ldots,j_l}, \tag{23}$$

where T_1 and T_2 are two tensors. If $\langle T_1, T_2\rangle = 0$ the two tensors are said orthogonal.

Now, we state the

Theorem 3 *Let $l \in \mathbb{N}$. Then, the vector space $\mathbb{M}_l$ given by Proposition 2 is generated by the components of the tensor $S_l(\vec{p})$ defined by*

$$S_l(\vec{p}) = \mathcal{T}_l(\vec{p}) + \sum_{k=1}^{[l/2]} \frac{(-m^2c^2)^k\,(l-k)!}{4^k(l-2k)!\,k!}\,\overline{\underbrace{g\otimes\ldots\otimes g}_{k \text{ times}}\otimes\mathcal{T}_{l-2k}(\vec{p})}, \tag{24}$$

and we have the following orthogonal decomposition

$$\mathbb{T}_l = S_l(\vec{p})\,\mathbb{R} \overset{\perp}{\oplus} \hat{\mathbb{T}}_l\,,$$

Proof. Let us denote by $\overline{\mathbb{M}}_l$ the vector space generated by the components of $S_l(\vec{p})$. For $k \ge 1$ and $l \ge 2$, we have

$$\begin{aligned} g_{i_1,i_2}\,\overline{\underbrace{g\otimes\ldots\otimes g}_{k \text{ times}}\otimes\mathcal{T}_{l-2k}(\vec{p})}^{i_1,i_2,i_3,\ldots,i_l} &= \frac{4k(l-k+1)}{l(l-1)}\,\overline{\underbrace{g\otimes\ldots\otimes g}_{k-1 \text{ times}}\otimes\mathcal{T}_{l-2k}(\vec{p})}^{i_3,\ldots,i_l} \\ &\quad + m^2c^2\,\frac{(l-2k)(l-2k-1)}{l(l-1)}\,\overline{\underbrace{g\otimes\ldots\otimes g}_{k \text{ times}}\otimes\mathcal{T}_{l-2-2k}(\vec{p})}^{i_3,\ldots,i_l}, \end{aligned} \tag{25}$$

for any $(i_3,\ldots,i_l) \in [\![0,3]\!]^{l-2}$. Consequently, this leads to

$$g_{i,j}\; S_l(\vec{p})^{i,j,i_1,\ldots,i_{l-2}} = 0,$$

for any $(i_1,\ldots,i_{l-2}) \in [\![0,3]\!]^{l-2}$ and then, $S_l(\vec{p})$ has at most $(l+1)^2$ independent components. Thus, $\dim\overline{\mathbb{M}}_l \le \dim(\mathbb{M}_l)$. But, by [3, Lemma 17.2.1], S_l is a four-tensor and therefore, $\overline{\mathbb{M}}_l$ satisfies condition (II). Moreover, the components of $S_l(\vec{p})$ are polynomials with degree l from $\mathcal{P}_l$. By Theorem 1, we conclude that $\overline{\mathbb{M}}_l = \mathbb{M}_l$. We deduce from (25) that the one-dimensional space generated by the tensor $S_l(\vec{p})$ is the orthogonal supplement of $\hat{\mathbb{T}}_l$ in $\mathbb{T}_l$ with respect to the Minkowski form (23). □

We now write down the moment spaces that arise in (21) for $l = 1$, $l = 2$, $l = 3$ and $l = 4$. Moreover we also consider here conditions (I) and (III).

Case $l = 1$ $\qquad \mathbb{M}_1 = \text{span}\,(\varepsilon, p^1, p^2, p^3)$.
This is the space generated by the four-vector $\vec{p}$. In order to satisfy condition (I), we add the mass and obtain the moment space $\tilde{\mathbb{P}}_1 = \text{span}\,(1, \vec{p})$. Whereas $\mathbb{M}_1$ is a 4-dimensional space, $\tilde{\mathbb{P}}_1$ is a 5-dimensional space. As stated in Section 1, $\tilde{\mathbb{P}}_1$ satisfies condition (III). The corresponding equations read

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f\,dp + \nabla_x \cdot \int_{\mathbb{R}^3} f\,v\,dp = 0, \tag{26}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f\,p\,dp + \nabla_x \cdot \int_{\mathbb{R}^3} f\,v\otimes p\,dp = 0, \tag{27}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f\,\varepsilon\,dp + c^2\,\nabla_x \cdot \int_{\mathbb{R}^3} f\,p\,dp = 0. \tag{28}$$

Case $l = 2$

$$\mathbb{M}_2 = \text{span}\,(\varepsilon p,\ (p^i p^j)_{i\neq j},\ m^2c^2 + |p|^2 + (p^1)^2,\ m^2c^2 + |p|^2 + (p^2)^2,\ m^2c^2 + |p|^2 + (p^3)^2).$$

The space $\mathbb{M}_2$ is a 9-dimensional space. Adding 1, p and ε, we obtain the 14-dimensional space $\tilde{\mathbb{P}}_2 = \text{span}\,(1,\ \vec{p},\ \vec{p}\otimes\vec{p})$ which satisfies conditions (I), (II) and (III). The space $\tilde{\mathbb{P}}_2$ leads to the following 14-moment system

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f dp + \nabla_x \cdot \int_{\mathbb{R}^3} f v dp = 0, \tag{29}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f p dp + \nabla_x \cdot \int_{\mathbb{R}^3} f v\otimes p dp = 0, \tag{30}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f \varepsilon dp + c^2 \nabla_x \cdot \int_{\mathbb{R}^3} f p dp = 0, \tag{31}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f \varepsilon p dp + c^2 \nabla_x \cdot \int_{\mathbb{R}^3} f p\otimes p dp = \int_{\mathbb{R}^3} Q_R(f,f)\varepsilon p dp, \tag{32}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f p\otimes p dp + \nabla_x \cdot \int_{\mathbb{R}^3} f v\otimes p\otimes p dp = \int_{\mathbb{R}^3} Q_R(f,f) p\otimes p dp. \tag{33}$$

Case $l = 3$

$$\begin{aligned}\mathbb{M}_3 = \text{span}\big(&\varepsilon(p^ip^j)_{i\neq j},\, p^1p^2p^3,\\ &\varepsilon(m^2c^2+|p|^2+3(p^1)^2),\, \varepsilon(m^2c^2+|p|^2+3(p^2)^2),\, \varepsilon(m^2c^2+|p|^2+3(p^3)^2),\\ &p_1(3m^2c^2+3|p|^2+(p^1)^2),\, p_1(m^2c^2+|p|^2+(p^2)^2),\, p_1(m^2c^2+|p|^2+(p^3)^2),\\ &p_2(m^2c^2+|p|^2+(p^1)^2),\, p_2(3m^2c^2+3|p|^2+(p^2)^2),\, p_2(m^2c^2+|p|^2+(p^3)^2),\\ &p_3(m^2c^2+|p|^2+(p^1)^2),\, p_3(m^2c^2+|p|^2+(p^2)^2),\, p_3(3m^2c^2+3|p|^2+(p^3)^2)\big).\end{aligned}$$

The dimension of $\mathbb{M}_3$ is thus 16. If we also consider the mass, momentum and energy, we obtain the space $\tilde{\mathbb{P}}_3 = \text{span}\,(1,\,\vec{p},\,\vec{p}\otimes\vec{p}\otimes\vec{p})$, whose dimension is 21. The space $\tilde{\mathbb{P}}_3$ satisfies conditions (I), (II) and (III) and leads to the moment system

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f dp + \nabla_x\cdot\int_{\mathbb{R}^3} f v dp = 0, \tag{34}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f p dp + \nabla_x\cdot\int_{\mathbb{R}^3} f v\otimes p dp = 0, \tag{35}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f \varepsilon dp + c^2\nabla_x\cdot\int_{\mathbb{R}^3} f p dp = 0, \tag{36}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f\varepsilon p\otimes p dp + c^2\nabla_x\cdot\int_{\mathbb{R}^3} f p\otimes p\otimes p dp = \int_{\mathbb{R}^3} Q_R(f,f)\varepsilon p\otimes p dp, \tag{37}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f p\otimes p\otimes p dp + \nabla_x\cdot\int_{\mathbb{R}^3} f v\otimes p\otimes p\otimes p dp = \int_{\mathbb{R}^3} Q_R(f,f) p\otimes p\otimes p dp. \tag{38}$$

Case $l = 4$

The space $\mathbb{M}_4$ is a 25-dimensional space. In order to satisfy condition (I), we add the mass, momentum and energy to $\mathbb{M}_4$ and obtain the space $\tilde{\mathbb{M}}_4$

$$\begin{aligned}\text{span}(&1,\, p,\, \varepsilon,\, \varepsilon p^1p^2p^3,\, (m^2c^2+|p|^2)^2+6(m^2c^2+|p|^2)(p^3)^2+(p^3)^4,\\ &(m^2c^2+|p|^2)^2+6(m^2c^2+|p|^2)(p^1)^2+6(p^1p^3)^2-(p^3)^4,\, (p^1)^4-6(p^1p^2)^2+(p^2)^4,\\ &(p^1)^4+6(m^2c^2+|p|^2)(p^1)^2-6(m^2c^2+|p|^2)(p^2)^2-(p^2)^4,\\ &(p^1p^2)^2-(m^2c^2+|p|^2)(p^3)^2+(m^2c^2+|p|^2)(p^2)^2-(p^3p^1)^2,\\ &(m^2c^2+|p|^2)(p^1)^2+(p^1p^3)^2-(m^2c^2+|p|^2)(p^2)^2-(p^2p^3)^2,\\ &\varepsilon p^1(m^2c^2+|p|^2+(p^1)^2),\, \varepsilon p^1(m^2c^2+|p|^2+3(p^2)^2),\, \varepsilon p^1(m^2c^2+|p|^2+3(p^3)^2),\\ &\varepsilon p^2(m^2c^2+|p|^2+3(p^1)^2),\, \varepsilon p^2(m^2c^2+|p|^2+(p^2)^2),\, \varepsilon p^2(m^2c^2+|p|^2+3(p^3)^2),\\ &\varepsilon p^3(m^2c^2+|p|^2+3(p^1)^2),\, \varepsilon p^3(m^2c^2+|p|^2+3(p^2)^2),\, \varepsilon p^3(m^2c^2+|p|^2+(p^3)^2),\\ &p^1p^2(3(m^2c^2+|p|^2)+(p^1)^2),\, p^1p^2(3(m^2c^2+|p|^2)+(p^2)^2),\, p^1p^2(m^2c^2+|p|^2+(p^3)^2),\\ &p^1p^3(3(m^2c^2+|p|^2)+(p^1)^2),\, p^1p^3(m^2c^2+|p|^2+(p^2)^2),\, p^1p^3(3(m^2c^2+|p|^2)+(p^3)^2),\\ &p^2p^3(m^2c^2+|p|^2+(p^1)^2),\, p^2p^3(3(m^2c^2+|p|^2)+(p^2)^2),\, p^2p^3(3(m^2c^2+|p|^2)+(p^3)^2)),\end{aligned}$$

that is an admissible space with degree 30 whereas the space $\tilde{\mathbb{P}}_4 = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$ has dimension 39.

Conclusion
The spaces $\mathbb{M}_l$ are strictly included in the spaces $\tilde{\mathbb{P}}_l$ defined by (18). When we also consider condition (I), we recover the spaces $\tilde{\mathbb{P}}_l$ and $\tilde{\mathbb{M}}_l$ introduced in Section 1. Contrary to the classical case, condition (III) had no consequence since the spaces we considered were already admissible.

The finite dimensional subspaces of $\mathbb{R}[\varepsilon, p^1, p^2, p^3]$ that satisfy condition (I), (II) and (III) are the vector sum of the spaces obtained as above for $l = 1, 2, 3, 4, \dots$ Hence, the admissible space with maximal degree 1, 2, 3 or 4 are

$$
\begin{array}{ll}
\text{maximal degree} = 1 & \mathbb{M} = \text{span}(1, \vec{p}), \\
\text{maximal degree} = 2 & \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}), \\
\text{maximal degree} = 3 & \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}), \\
 & \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}), \\
\text{maximal degree} = 4 & \mathbb{M} = \mathbb{M}_4 \oplus \text{span}(1, \vec{p}), \\
 & \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p}), \\
 & \mathbb{M} = \text{span}(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p}),
\end{array}
$$

which have respectively dimension 5, 14, 21, 30, 30, 39 and 55.

3 Classical limit

The classical limit consists in considering velocities v that are much smaller than the speed of light c. This amounts to let $v/c \to 0$. The equations should be rescaled but, for the sake of clarity, we keep our notations and let $c \to +\infty$. From (3), we deduce that

$$\varepsilon = mc^2 + \frac{m|v|^2}{2} + \frac{3m|v|^4}{8c^2} + O\left(\frac{1}{c^4}\right) \qquad \text{and} \qquad p = mv + \frac{mv|v|^2}{2c^2} + O\left(\frac{1}{c^4}\right). \tag{39}$$

It implies that

$$f(t,x,p) = \frac{1}{m^3}\, f_c(t,x,v) + O\left(\frac{1}{c^2}\right) \qquad \text{and} \qquad dp = m^3\, dv + O\left(\frac{1}{c^2}\right). \tag{40}$$

We denote here by f_c the distribution function in the classical case.

3.1 System $(1, \vec{p})$

We consider the classical limit of (26)-(28). With (39) and (40), equations (26) and (27) become

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv + O\left(\frac{1}{c^2}\right) &= 0, \\ \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v dv + O\left(\frac{1}{c^2}\right) &= 0. \end{aligned}$$

We thus obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv = 0, \tag{41}$$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v dv = 0. \tag{42}$$

Moreover, equation (28) reads

$$mc^2\left(\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv\right) + \frac{m}{2}\left(\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c |v|^2 v dv\right) + O\left(\frac{1}{c^2}\right) = 0,$$

which, with (41), leads to

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c |v|^2 v dv + O\left(\frac{1}{c^2}\right) = 0,$$

that is,

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c |v|^2 v dv = 0.$$

We finally obtain the following system

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v dv &= 0, \\ \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v dv &= 0, \\ \frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c |v|^2 v dv &= 0, \end{aligned}$$

that is the equations associated to the moment space $\mathrm{span}(1, v, |v|^2)$. This space is invariant under any rotation and translation, and it is an admissible moment space.

3.2 System $(1, \vec{p}, \vec{p} \otimes \vec{p})$

We consider the classical limit of (29)-(33). We pass to the limit $c \to +\infty$ in (29)-(31) as we did for (26)-(28). With (39) and (40), equation (33) becomes

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f_c \, v \otimes v \, dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c \, v \otimes v \otimes v \, dv + O\left(\frac{1}{c^2}\right) = \frac{1}{m^2} \int_{\mathbb{R}^3} Q_R(f, f) \, p \otimes p \, dp.$$

We still need to pass to the limit in the collision kernel $Q_R(f, f)$. We deduce from (39) that (8) and (9) read

$$v_M = |v - v_*| + O\left(\frac{1}{c^2}\right) \qquad \text{and} \qquad \sigma = \sigma_c + O\left(\frac{1}{c^2}\right),$$

where

$$\sigma_c = \left(\frac{q q_*}{8\mu |v - v_*|^2 \pi \varepsilon_0}\right)^2 \frac{1}{\sin^4(\theta/2)} \qquad \text{with} \qquad \mu = \frac{m m_*}{m + m_*}.$$

Moreover, (40) implies that

$$f(p^\natural) f(p_*^\natural) - f(p) f(p_*) = \frac{1}{m^3 m_*^3} \left(f_c(v^\natural) f_c(v_*^\natural) - f_c(v) f_c(v_*) \right) + O\left(\frac{1}{c^2}\right),$$

where the velocities $v^\natural$ and $v_*^\natural$ are solutions to the conservation laws of momentum and energy

$$m v + m_* v_* = m v^\natural + m_* v_*^\natural \qquad \text{and} \qquad m|v|^2 + m_*|v_*|^2 = m|v^\natural|^2 + m_*|v_*^\natural|^2.$$

We thus obtain that

$$Q_R(f, f) = \frac{1}{m^3} \, Q_C(f_c, f_c) + O\left(\frac{1}{c^2}\right), \tag{43}$$

where Q_C denotes the classical collision kernel

$$Q_C(f_c, f_c)(t, x, v) = \iint_{\mathbb{S}^2 \times \mathbb{R}^3} \sigma_c \, |v - v_*| \, (f_c(v^\natural) f_c(v_*^\natural) - f_c(v) f_c(v_*)) \, dv_* \, d\omega.$$

From (43), we deduce that (33) reads

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f_c v \otimes v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v \otimes v dv + O\left(\frac{1}{c^2}\right) = \int_{\mathbb{R}^3} Q_C(f_c, f_c) v \otimes v dv + O\left(\frac{1}{c^2}\right).$$

We finally get

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f_c v \otimes v dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c v \otimes v \otimes v dv = \int_{\mathbb{R}^3} Q_C(f_c, f_c) v \otimes v dv.$$

Similarly to (28), equation (32) becomes

$$m^2c^2\left(\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c\, v\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c\, v\otimes v\, dv\right)$$
$$+\,\frac{m^2}{2}\left(\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c\, |v|^2\, v\, dv + \nabla_x \cdot \int_{\mathbb{R}^3} f_c\, |v|^2\, v\otimes v\, dv\right) + O\left(\frac{1}{c^2}\right) = \int_{\mathbb{R}^3} Q_R(f,f)\,\varepsilon\, p\, dp.$$

But,

$$\int_{\mathbb{R}^3} Q_R(f,f)\varepsilon p dp = \frac{m^2}{2}\int_{\mathbb{R}^3} Q_C(f_c,f_c)|v|^2 v dv + O\left(\frac{1}{c^2}\right),$$

whence, with (42),

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c|v|^2 v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c|v|^2 v\otimes v dv + O\left(\frac{1}{c^2}\right) = \int_{\mathbb{R}^3} Q_C(f_c,f_c)|v|^2 v dv + O\left(\frac{1}{c^2}\right).$$

Finally, system (29)-(33) becomes, letting $c \to +\infty$,

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c|v|^2 v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c|v|^2 v\otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c,f_c)|v|^2 v dv,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v dv.
\end{aligned}$$

We thus obtain the moment space $\mathrm{span}(1, v, v\otimes v, v|v|^2)$, that is the Grad 13-moment system. The Grad 13-moment system is therefore compatible with the relativistic system. This space is stable under any rotation and translation. However, it is not an admissible space (in the sense of Levermore). Here the limit system has dimension 13 whereas the system (29)-(33) has dimension 14 because the equation involving $|v|^2$ is obtained twice.

3.3 System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p})$

We have already passed to the limit in (34)-(36). With (39), (40) and (43), equations (37) and (38) lead to

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv + O\left(\frac{1}{c^2}\right) = \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v dv + O\left(\frac{1}{c^2}\right),$$

and

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v\otimes v dv + O\left(\frac{1}{c^2}\right)$$
$$= \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v\otimes v dv + O\left(\frac{1}{c^2}\right).$$

Thus, letting $c\to+\infty$, we obtain the following equations

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v dv,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v\otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v\otimes v dv,
\end{aligned}$$

that is the moment system corresponding to $(1, v, v\otimes v, v\otimes v\otimes v)$. This system is invariant under any rotation and translation but is not an admissible system (in the sense of Levermore). This space has dimension 20.

3.4 System $(1,\vec{p},\vec{p}\otimes\vec{p},\vec{p}\otimes\vec{p}\otimes\vec{p})$

We deduce from the above calculations that passing to the limit in the moment system associated to $(1,\vec{p},\vec{p}\otimes\vec{p},\vec{p}\otimes\vec{p}\otimes\vec{p})$ leads to

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v dv &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v dv,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c v\otimes v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c v\otimes v\otimes v\otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c,f_c) v\otimes v\otimes v dv,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_c |v|^2 v\otimes v dv + \nabla_x\cdot\int_{\mathbb{R}^3} f_c |v|^2 v\otimes v\otimes v dv &= \int_{\mathbb{R}^3} Q_C(f_c,f_c) |v|^2 v\otimes v dv,
\end{aligned}$$

that is the moment system corresponding to $(1, v, v\otimes v, v\otimes v\otimes v, |v|^2 v\otimes v)$. This system is invariant under any rotation and translation. Moreover, it is an admissible system whose dimension is 26.

3.5 System $\tilde{\mathbb{M}}_4$

The system $\tilde{\mathbb{M}}_4$ consists of 30 independent moments. Consequently, we do not write down all the equations and we do not give all the details of the passage to the limit. The different steps are described below.

- As previously, the moment system $(1, p, \varepsilon)$ leads to the moment system $(1, v, |v|^2)$.
- From the set of moments of the form $p^i p^j (3(m^2c^2 + |p|^2) + (p^i)^2)$ and $p^i p^j (m^2c^2 + |p|^2 + (p^k)^2)$, we obtain the moments $v_i v_j$, $v_i v_j (v_i^2 - v_j^2)$ and $v_i v_j (v_i - 3v_k)$ for $i \neq j$, $k \neq i$ and $k \neq j$.
- Passing to the limit in the set of moments of the form $\varepsilon p^i (m^2c^2 + |p|^2 + (p^i)^2)$ and $\varepsilon p^i (m^2c^2 + |p|^2 + 3(p^k)^2)$, we get the moments $|v|^2 v_i$ and $v_i(v_i^2 - 3v_k^2)$ for $i \neq k$.
- The moment $\varepsilon p^1 p^2 p^3$ leads to the moment $v_1 v_2 v_3$.
- The six remaining moments lead to the moments $v_i^2 - v_j^2$ and $v_i^4 + 3v_j^2 v_k^2 - 3v_i^2 v_j^2 - 3v_i^2 v_k^2$, for $i \neq j$, $i \neq k$, and $j \neq k$.

Summarizing, the limit system is the following 29-dimensional space

$$\begin{aligned}\mathrm{span}\,(&1, v, v\otimes v, v\otimes v\otimes v,\ v_1v_2(v_1^2 - v_2^2),\ v_1v_2(v_1^2 - 3v_3^2),\ v_1v_3(v_1^2 - v_3^2),\\ &v_1v_3(v_1^2 - 3v_2^2),\ v_2v_3(v_2^2 - v_3^2),\ v_2v_3(v_2^2 - 3v_1^2),\ v_1^4 + 3v_2^2v_3^2 - 3v_1^2v_2^2 - 3v_1^2v_3^2,\\ &v_2^4 + 3v_1^2v_3^2 - 3v_2^2v_3^2 - 3v_1^2v_2^2,\ v_3^4 + 3v_1^2v_2^2 - 3v_1^2v_3^2 - 3v_2^2v_3^2),\end{aligned}$$

which also reads

$$\mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v - 6|v|^2/7\overline{I_3\otimes v\otimes v} + 3|v|^4/35 I_3\otimes I_3),$$

where $\overline{T}$ denotes the symmetric part of the tensor T. The obtained space is invariant under any rotation and any translation but it is non admissible.

3.6 System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$

Passing to the limit $c \to +\infty$ as in the previous sections, we get the moment space

$$\mathrm{span}\,(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v),$$

whose dimension is 35. This space is stable under any rotation and translation and it is admissible.

3.7 System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$

Letting $c \to +\infty$, we get the moment space

$$\mathrm{span}\,(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v, |v|^2 v\otimes v\otimes v).$$

This space is non admissible and has dimension 45. It is invariant under any rotation and any translation.

3.8 Conclusion

Passing to the limit $c \to +\infty$ on admissible relativistic systems, we have obtained admissible classical spaces as $(1, v, |v|^2)$ and $(1, v, v \otimes v, v \otimes v \otimes v, |v|^2 v \otimes v)$, but also non admissible spaces as $(1, v, v \otimes v, v|v|^2)$ and $(1, v, v \otimes v, v \otimes v \otimes v)$. Let us summarize the obtained limit systems:

Relativistic system	Limit system
$(1, \vec{p})$	$(1, v, \|v\|^2)$
$(1, \vec{p}, \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, \|v\|^2 v)$
$(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v)$
$(1, \vec{p}, \vec{p} \otimes \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v, \|v\|^2 v \otimes v)$
$\mathbb{M}_4 \oplus \mathrm{span}(1, \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v,$
	$v \otimes v \otimes v \otimes v - 6\|v\|^2/7\overline{I_3 \otimes v \otimes v} + 3\|v\|^4/35 I_3 \otimes I_3)$
$(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v, v \otimes v \otimes v \otimes v)$
$(1, \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p}, \vec{p} \otimes \vec{p} \otimes \vec{p} \otimes \vec{p})$	$(1, v, v \otimes v, v \otimes v \otimes v, v \otimes v \otimes v \otimes v, \|v\|^2 v \otimes v \otimes v)$

By letting $c \to +\infty$ in the relativistic moment spaces, we do not recover all the classical moment spaces, for instance, we did not get $(1, v, v \otimes v)$. Since classical mechanics is considered as an approximation of relativistic mechanics as $c \to +\infty$, it could be sensible to choose as moment spaces in the classical case only the admissible moment spaces that can be obtained as a limit of relativistic ones.

4 The ultra-relativistic case

The ultra-relativistic limit corresponds to the case when the total energy ε of a particle is much larger than its rest energy $m\,c^2$. As for the classical limit, the equations should be rescaled but we do not want to get the reader confused with non necessary details. Consequently, we keep our notations and let m tend to 0. Formulas (4) read

$$\varepsilon = c|p| + O(m^2) \qquad \text{and} \qquad v = c\frac{p}{|p|} + O(m^2). \tag{44}$$

We deduce then that (8) and (9) read

$$v_M = v_{M\,ur} + O\left(m^2\right) \qquad \text{and} \qquad \sigma = \sigma_{ur} + O\left(m^2\right),$$

where

$$v_{M\,ur} = |v_{rel}|\left(1 - \frac{p \cdot p_*}{|p||p_*|}\right) \qquad \text{and} \qquad \sigma_{ur} = \left(\frac{qq_*}{4\pi\varepsilon_0}\right)^2 \frac{1 + \cos^4(\theta/2)}{8c^2|\overline{p}|^2 \sin^4(\theta/2)}.$$

We thus obtain that

$$Q_R(f,f) = Q_{ur}(f_{ur}, f_{ur}) + O\left(m^2\right), \tag{45}$$

where f_{ur} denotes the ultra-relativistic distribution function and Q_{ur} the ultra-relativistic collision kernel

$$Q_{ur}(f_{ur}, f_{ur})(t,x,v) = \iint_{\mathbb{S}^2\times\mathbb{R}^3} v_{M\,ur}\sigma_{ur}\left(f_{ur}(p^\natural)f_{ur}(p_*^\natural) - f_{ur}(p)f_{ur}(p_*)\right) dp_*\, d\omega.$$

System $(1, \vec{p})$
With (44), equations (26)-(28) lead to

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}dp &= 0, && (46)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}p dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}\otimes p dp &= 0, && (47)\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p| dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}p dp &= 0, && (48)
\end{aligned}$$

that is the 5-moment system associated to $(1, p, |p|)$.

System $(1, \vec{p}, \vec{p}\otimes\vec{p})$
By (44) and (45), equations (29)-(33) read

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}dp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}p dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}\otimes p dp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p| dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}p dp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p|p dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}p\otimes p dp &= \int_{\mathbb{R}^3} Q_{ur}(f_{ur}, f_{ur})|p|p dp,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}p\otimes p dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}\otimes p\otimes p dp &= \int_{\mathbb{R}^3} Q_{ur}(f_{ur}, f_{ur})p\otimes p dp.
\end{aligned}$$

We thus obtain the 14-dimensional system $(1, p, |p|, p|p|, p\otimes p)$.

System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p})$
In the ultra-relativistic case, equations (34)-(38) become

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}\frac{p}{|p|}dp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}p dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}\frac{p\otimes p}{|p|}dp &= 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p|dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}p dp &= 0,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}|p|p\otimes p dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}p\otimes p\otimes p dp &= \int_{\mathbb{R}^3} Q_{ur}(f_{ur},f_{ur})|p|p\otimes p dp,\\
\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f_{ur}p\otimes p\otimes p dp + c\nabla_x\cdot\int_{\mathbb{R}^3} f_{ur}\frac{p\otimes p\otimes p\otimes p}{|p|}dp &= \int_{\mathbb{R}^3} Q_{ur}(f_{ur},f_{ur})p\otimes p\otimes p dp.
\end{aligned}$$

We thus obtain the system $(1, p, |p|, |p|p\otimes p, p\otimes p\otimes p)$, whose dimension is 21.

System $(1, \vec{p}, \vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p})$
Letting $m\to 0$, we obtain the moment space

$$\text{span}\,(1,\ p,\ |p|,\ |p|p,\ p\otimes p,\ |p|p\otimes p,\ p\otimes p\otimes p),$$

whose dimension is 30.

System $\tilde{\mathbb{M}}_4$
In the ultra-relativistic case, the space $\tilde{\mathbb{M}}_4$ becomes

$$\begin{aligned}
\text{span}(&1,\ p,\ |p|,\ |p|p^1p^2p^3,\ |p|^4+6|p|^2(p^3)^2+(p^3)^4,\ |p|^4+6|p|^2(p^1)^2+6(p^1p^3)^2-(p^3)^4,\\
&(p^1p^2)^2-|p|^2(p^3)^2+|p|^2(p^2)^2-(p^3p^1)^2,\ |p|^2(p^1)^2+(p^1p^3)^2-|p|^2(p^2)^2-(p^2p^3)^2,\\
&(p^1)^4-6(p^1p^2)^2+(p^2)^4,\ (p^1)^4+6|p|^2(p^1)^2-6|p|^2(p^2)^2-(p^2)^4,\ |p|(p^1)^3,\ |p|p^1(p^2)^2,\\
&|p|p^1(p^3)^2,\ |p|p^2(p^1)^2,\ |p|(p^2)^3,\ |p|p^2(p^3)^2,\ |p|p^3(p^1)^2,\ |p|p^3(p^2)^2,\ |p|(p^3)^3,\ (p^1)^3p^2,\\
&p^1(p^2)^3,\ p^1p^2(p^3)^2,\ (p^1)^3p^3,\ p^1(p^2)^2p^3,\ p^1(p^3)^3,\ (p^1)^2p^2p^3,\ (p^2)^3p^3,\ p^2(p^3)^3).
\end{aligned}$$

System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$
Passing to the limit $m\to 0$, we get the moment space

$$\text{span}\,(1,\ p,\ |p|,\ |p|p,\ p\otimes p,\ |p|p\otimes p\otimes p,\ p\otimes p\otimes p\otimes p),$$

whose dimension is 39.

System $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$
In the ultra-relativistic case, the moment space $(1, \vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}, \vec{p}\otimes\vec{p}\otimes\vec{p}\otimes\vec{p})$ leads to

$$\text{span}\,(1,\ p,\ |p|,\ |p|p,\ p\otimes p,\ |p|p\otimes p,\ p\otimes p\otimes p,\ |p|p\otimes p\otimes p,\ p\otimes p\otimes p\otimes p).$$

This space has dimension 55.

5 Moment closure problem

5.1 The maximum entropy principle

Up to now, we have determine the moment spaces $\mathbb{M}$ that could be used to derive moment system, as well in the relativistic case than in the ultra-relativistic case. The moment system is then obtained by multiplying the Boltzmann equation by a basis $m = (m_i)_{1\leq i\leq N}$ of $\mathbb{M}$ and integrating with respect to the momentum variable. However, the obtained moment system is not closed unless a distribution function is specified. A usual strategy consists in closing this system using the function that solves the entropy minimization problem. Given $M \in \mathbb{R}^N$, we close thanks to the distribution f that realizes the following minimum

$$\min\left\{S(f) = \int_{\mathbb{R}^3} (f\ln f - f)\, dp, \qquad \int_{\mathbb{R}^3} f(p)\, m(p,\varepsilon(p))\, dp = M\right\}. \tag{49}$$

It is of course not warranted that this problem has a solution. Indeed, the vector $M \in \mathbb{R}^N$ needs to satisfy some constraints in order that there exists a distribution f such that

$$\int_{\mathbb{R}^3} f(p)\, m(p,\varepsilon(p))\, dp = M. \tag{50}$$

Formally, the method of the Lagrange multipliers imply that the solution f of the entropy minimization problem (49) satisfies

$$f(t,x,p) = \exp\big(\alpha(t,x)\cdot m(p,\varepsilon(p))\big),$$

where the coefficient $\alpha \in \mathbb{R}^N$ is determined by the constraint (50). Consequently, the main point is to know whether there exist exponential densities that satisfy (50). This problem has been answered in the classical case by Junk [11] and Schneider [19], who assumed that there exists one moment of the basis that grows faster than the others at infinity. This assumption is fulfilled neither by the relativistic moment spaces nor by the ultra-relativistic moment spaces. It would be interesting to see if the results of Junk and Schneider could however be extended to these cases. Of course, the ultra-relativistic case should be easier to handle because the corresponding energy $\overline{\varepsilon} = c|p|$ is much simpler than the relativistic one. Consequently, some calculations might be explicit. This problem is not considered herein.

In the relativistic and ultra-relativistic cases, the moment realizability problem has already been solved for the moments $(1, p, \varepsilon(p))$ and $(1, p, \overline{\varepsilon}(p))$. Consequently, for these moment systems, the closure by the entropy minimization principle can be carried out. We summarize the main ideas below.

5.2 The relativistic case

We proceed here below to the closure of the 5 moment system (26)-(28). By [7, Theorem 3.15.3] and [5, Theorem 2.1], there exists a solution to the problem of minimizing the entropy at fixed mass n, momentum P and energy W if and only if n, P and W satisfy $m^2c^2n^2 + |P|^2 \leq W^2/c^2$ and this solution is uniquely determined. This solution is the relativistic Maxwellian of the form (11) that satisfies

$$n = \int_{\mathbb{R}^3} \mathcal{M}(p)\, dp, \qquad P = \int_{\mathbb{R}^3} \mathcal{M}(p)\, p\, dp \qquad \text{and} \qquad W = \int_{\mathbb{R}^3} \mathcal{M}(p)\, \varepsilon(p)\, dp.$$

The closure of the system (26)-(28) thanks to this Maxwellian enables us to compute the fluxes $\int_{\mathbb{R}^3} f v dp$ and $\int_{\mathbb{R}^3} f v \otimes p dp$ in terms of n, P and W. We obtain the relativistic hydrodynamic equations (see [13, 15])

$$\frac{\partial}{\partial t}\begin{pmatrix} n \\ P \\ W \end{pmatrix} + \nabla_x \cdot \begin{pmatrix} n\, u \\ P \otimes u + \mathcal{P}\, \gamma_u^{-1}\, Id \\ W\, u + \mathcal{P}\, \gamma_u^{-1}\, u \end{pmatrix} = 0,$$

where $\mathcal{P}$, W and P are related to T, u and n by

$$\mathcal{P} = n\, k\, T, \qquad W = \gamma_u \left(n\, e_0(T) + \frac{|u|^2}{c^2} \mathcal{P} \right) \qquad \text{and} \qquad P = \gamma_u\, \frac{u}{c^2} \big(n\, e_0(T) + \mathcal{P} \big).$$

Here, we denote respectively by u, $\mathcal{P}$, T and e_0 the velocity, the stress, the temperature and the proper internal energy of the fluid. The constant k is Boltzmann's constant.

5.3 The ultra-relativistic case

We close here the system (46)-(48) thanks to the Maxwellian that minimizes the entropy at fixed mass $\tilde{n}$, momentum $\tilde{P}$ and energy $\tilde{W}$. The proof of [5, Theorem 2.1] can be extended to the ultra-relativistic case and there exists a solution to this problem if and only if $\tilde{P}$ and $\tilde{W}$ satisfy $|\tilde{P}| \leq \tilde{W}/c$, this solution being uniquely determined. This solution is the ultra-relativistic Maxwellian of the form

$$\tilde{\mathcal{M}}(p) = A\, \exp\big(-\beta^0\, |p| + \beta \cdot p \big) \qquad \text{with} \qquad A \in \mathbb{R}_+, \beta^0 \in \mathbb{R}_+,\, \beta \in \mathbb{R}^3,$$

that satisfies

$$\tilde{n} = \int_{\mathbb{R}^3} \tilde{\mathcal{M}}(p)\, dp, \qquad \tilde{P} = \int_{\mathbb{R}^3} \tilde{\mathcal{M}}(p)\, p\, dp \qquad \text{and} \qquad \tilde{W} = c \int_{\mathbb{R}^3} \tilde{\mathcal{M}}(p)\, |p|\, dp.$$

We obtain the ultra-relativistic hydrodynamic equations (see [12])

$$\frac{\partial}{\partial t}\begin{pmatrix} \tilde{n} \\ \tilde{P} \\ \tilde{W} \end{pmatrix} + \nabla_x \cdot \begin{pmatrix} \tilde{n}\, \tilde{u} \\ \tilde{P} \otimes \tilde{u} + \tilde{\mathcal{P}}\, \gamma_{\tilde{u}}^{-1}\, Id \\ \tilde{W}\, \tilde{u} + \tilde{\mathcal{P}}\, \gamma_{\tilde{u}}^{-1}\, \tilde{u} \end{pmatrix} = 0,$$

where $\tilde{\mathcal{P}}$, $\tilde{W}$ and $\tilde{P}$ are related to $\tilde{T}$, $\tilde{u}$ and $\tilde{n}$ by

$$\tilde{\mathcal{P}} = \tilde{n}\, k\, \tilde{T}, \qquad \tilde{W} = \gamma_{\tilde{u}} \left(3 + \frac{|\tilde{u}|^2}{c^2} \right) \tilde{\mathcal{P}} \qquad \text{and} \qquad \tilde{P} = 4\, \gamma_{\tilde{u}}\, \frac{\tilde{u}}{c^2}\, \tilde{\mathcal{P}}.$$

Here, we denote respectively by $\tilde{u}$, $\tilde{\mathcal{P}}$ and $\tilde{T}$ the velocity, the stress and the temperature of the fluid. The constant k is Boltzmann's constant.

6 Proof of Theorem 1

This section is based on the representation theory of Lie groups and Lie algebras. Therefore, we refer to [6, 10] for further information. The group $SO(1,3)_e$ is a matrix Lie group. We point out that Lie algebras are essential for the study of matrix Lie groups because they have the advantage of being vector spaces and thus allow the use of linear algebra tools. The Lie algebra associated to $SO(1,3)_e$ reads

$$so_{\mathbb{R}}(1,3) = \left\{X \in M(4,\mathbb{R}); gX^T + Xg = 0\right\},$$

where X^T denotes the matrix transpose of X. Let $so_{\mathbb{C}}(1,3)$ be the complexification of $so_{\mathbb{R}}(1,3)$ (see [10, Definition 2.43]),

$$so_{\mathbb{C}}(1,3) = \left\{X \in M(4,\mathbb{C}); gX^T + Xg = 0\right\}.$$

We denote by $\mathbb{C}_n[y_0,y_1,y_2,y_3]$ the set of complex homogeneous polynomials with degree n and consider the following representation of $SO(1,3)_e$:

$$\begin{array}{rcl} \tilde{\varphi} : SO(1,3)_e & \longrightarrow & GL(\mathbb{C}_n[y_0,y_1,y_2,y_3]) \\ L & \longmapsto & \{R(y_0,y_1,y_2,y_3) \longmapsto R(L^{-1}(y_0,y_1,y_2,y_3))\} \end{array}$$

By [10, Proposition 4.4], the representation $(\tilde{\varphi}, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ of $SO(1,3)_e$ induces a unique representation $(\Phi, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ of $so_{\mathbb{R}}(1,3)$, which is defined by

$$\Phi(Z) = \frac{d}{dt}\tilde{\varphi}(e^{tZ})\Big|_{t=0}, \qquad Z \in so_{\mathbb{R}}(1,3).$$

By [10, Proposition 4.6], this finite dimensional complex representation of $so_{\mathbb{R}}(1,3)$ may be uniquely extended to a complex representation $(\tilde{\Phi}, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ of $so_{\mathbb{C}}(1,3)$, given by

$$\tilde{\Phi}(Z) = \Phi(Z_1) + i\Phi(Z_2), \qquad Z = Z_1 + iZ_2 \in so_{\mathbb{C}}(1,3), \quad Z_1, Z_2 \in so_{\mathbb{R}}(1,3).$$

The representation theory of Lie algebras implies, thanks to the highest weight theory, that the following theorem holds:

Theorem 4 *The representation $(\tilde{\Phi}, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ of $so_{\mathbb{C}}(1,3)$ is not irreducible. More precisely, we have the following decomposition:*

$$\mathbb{C}_n[y_0,y_1,y_2,y_3] = \bigoplus_{j=0}^{[n/2]} \Gamma^{(n)}_{n-2j,0}, \tag{51}$$

where $\Gamma^{(n)}_{n-2j,0}$ *is a subspace of* $\mathbb{C}_n[y_0,y_1,y_2,y_3]$ *generated by*

$$\left((y_0^2-y_1^2-y_2^2-y_3^2)^j \sum_{m=\max(l-k,0)}^{\min(l,n-2j-k)} \frac{(n-2j-k)!}{(n-2j-k-m)!}\frac{k!}{(k-l+m)!}\binom{l}{m}\right.$$
$$\left.(y_1-iy_2)^{n-2j-k-m}(y_0+y_3)^m(y_0-y_3)^{k-l+m}(y_1+iy_2)^{l-m}\right)_{0\le k,\, l\le n-2j}. \quad (52)$$

Each representation $(\tilde{\Phi}_{|\Gamma^{(n)}_{n-2j,0}}, \Gamma^{(n)}_{n-2j,0})$ *is irreducible.*

It then follows easily that

Theorem 5 *The representation* $(\tilde{\varphi}, \mathbb{C}_n[y_0,y_1,y_2,y_3])$ *of* $SO(1,3)_e$ *is not irreducible. More precisely, (51) holds and each* $(\tilde{\varphi}_{|\Gamma^{(n)}_{n-2j,0}}, \Gamma^{(n)}_{n-2j,0})$ *is an irreducible representation.*

Proof. We have to show that each $\Gamma^{(n)}_{n-2j,0}$ is stable under $SO(1,3)_e$. Since $SO(1,3)_e$ is generated by the matrices

$$R_1(t)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos t&\sin t\\0&0&-\sin t&\cos t\end{pmatrix}, \qquad R_2(t)=\begin{pmatrix}1&0&0&0\\0&\cos t&0&\sin t\\0&0&1&0\\0&-\sin t&0&\cos t\end{pmatrix},$$

$$R_3(t)=\begin{pmatrix}1&0&0&0\\0&\cos t&\sin t&0\\0&-\sin t&\cos t&0\\0&0&0&1\end{pmatrix}, \qquad L_1(t)=\begin{pmatrix}\cosh t&\sinh t&0&0\\\sinh t&\cosh t&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

$$L_2(t)=\begin{pmatrix}\cosh t&0&\sinh t&0\\0&1&0&0\\\sinh t&0&\cosh t&0\\0&0&0&0\end{pmatrix}, \qquad L_3(t)=\begin{pmatrix}\cosh t&0&0&\sinh t\\0&1&0&0\\0&0&1&0\\\sinh t&0&0&\cosh t\end{pmatrix},$$

for $t\in\mathbb{R}$, it suffices to show that the vector space $\Gamma^{(n)}_{n-2j,0}$ is stable under any $\tilde{\varphi}(R_k(t))$ and $\tilde{\varphi}(L_k(t))$, for every $t\in\mathbb{R}$. The space $so_{\mathbb{C}}(1,3)$ is generated by

$$R_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}, \quad R_2=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}, \quad R_3=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}, \quad (53)$$

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (54)$$

Consequently, each $\Gamma^{(n)}_{n-2j,0}$ is stable under $\Phi(R_k)$ and $\Phi(L_k)$, $1 \leq k \leq 3$. But, we have

$$R_k(t) = \exp(tR_k) \qquad \text{and} \qquad L_k(t) = \exp(tL_k), \qquad t \in \mathbb{R},$$

and, by [10, Proposition 4.4],

$$\tilde{\varphi}\left(e^X\right) = e^{\Phi(X)}, \qquad X \in so_{\mathbb{R}}(1,3).$$

We thus deduce that $\Gamma^{(n)}_{n-2j,0}$ is stable under any $\tilde{\varphi}(R_k(t))$ and $\tilde{\varphi}(L_k(t))$, for every $t \in \mathbb{R}$.

Since $SO(1,3)_e$ is a connected matrix Lie group, the representation $(\tilde{\varphi}_{|\Gamma^{(n)}_{n-2j,0}}, \Gamma^{(n)}_{n-2j,0})$ of $SO(1,3)_e$ is irreducible by [10, Proposition 4.5] and [10, Proposition 4.6]. □

It remains now to consider the case of real polynomials. We denote by $\mathbb{R}_n[y_0, y_1, y_2, y_3]$ the set of real homogeneous polynomials with degree n and consider the representation $(\varphi_{|\mathbb{R}_n[y_0,y_1,y_2,y_3]}, \mathbb{R}_n[y_0, y_1, y_2, y_3])$ of $SO(1,3)_e$, where φ is defined by (19).

Theorem 6 *The representation $(\varphi_{|\mathbb{R}_n[y_0,y_1,y_2,y_3]}, \mathbb{R}_n[y_0, y_1, y_2, y_3])$ of $SO(1,3)_e$ is not irreducible. We have the following decomposition:*

$$\mathbb{R}_n[y_0, y_1, y_2, y_3] = \bigoplus_{j=0}^{[n/2]} \tilde{\Gamma}^{(n)}_{n-2j,0}, \tag{55}$$

where the space $\tilde{\Gamma}^{(n)}_{n-2j,0}$ is generated by the real parts and the imaginary parts of

$$(y_0^2 - y_1^2 - y_2^2 - y_3^2)^j \sum_{m=\max(q-r,0)}^{q} \frac{(n-2j-r)!r!}{(n-2j-r-m)!(r-q+m)!} \binom{q}{m}$$
$$(y_0+y_3)^m (y_0-y_3)^{r-q+m} (y_1+iy_2)^{n-2j-r-m} (y_1-iy_2)^{q-m}, \tag{56}$$

for $q, r \in [\![0, n-2j]\!]$, $q + r \leq n-2j$. The subrepresentations $(\varphi_{|\tilde{\Gamma}^{(n)}_{n-2j,0}}, \tilde{\Gamma}^{(n)}_{n-2j,0})$ are irreducible.

Proof. Let $(q,r) \in \mathbb{N}^2$ such that $q + r < n - 2j$. Choosing $(k,l) = (r,q)$ and $(k,l) = (n-2j-q, n-2j-r)$, we deduce that the complex basis (52) can be replaced by the real parts and imaginary parts of (56). We denote by $\tilde{\Gamma}^{(n)}_{n-2j,0}$ the real vector space generated by the real parts and the imaginary parts of (56). It follows from (51) that (55) holds.

We still have to check that each $\tilde{\Gamma}^{(n)}_{n-2j,0}$ is stable under $SO(1,3)_e$. By Theorem 5, we already know that the complexification $\Gamma^{(n)}_{n-2j,0}$ of $\tilde{\Gamma}^{(n)}_{n-2j,0}$ is stable under $SO(1,3)_e$.

Since the proper Lorentz group $SO(1,3)_e$ is composed of real matrices, we deduce that the real vector space $\tilde{\Gamma}^{(n)}_{n-2j,0}$ is stable under $SO(1,3)_e$.

Each representation $(\varphi_{|\tilde{\Gamma}^{(n)}_{n-2j,0}}, \tilde{\Gamma}^{(n)}_{n-2j,0})$ of $SO(1,3)_e$ is irreducible. Indeed, if there is a subspace V of $\tilde{\Gamma}^{(n)}_{n-2j,0}$ stable under $SO(1,3)_e$, then the complexification $V+iV$ of V is a subspace of $\Gamma^{(n)}_{n-2j,0}$ stable under $SO(1,3)_e$. Since $\Gamma^{(n)}_{n-2j,0}$ is irreducible, we deduce that either $V=\{0\}$, or $V=\tilde{\Gamma}^{(n)}_{n-2j,0}$. □

We now consider the representation (19) of $SO(1,3)_e$ and we prove Theorem 1. It follows from (55) that we have the following decomposition of $(\varphi, \mathcal{P}_n)$ as a direct sum of irreducible representations

$$\mathcal{P}_n = \bigoplus_{k=0}^{n} \bigoplus_{j=0}^{[k/2]} \tilde{\Gamma}^{(k)}_{k-2j,0}, \tag{57}$$

where $\tilde{\Gamma}^{(k)}_{k-2j,0}$ is the real vector space given by Theorem 6. The Schur Lemma (see [10, Theorem 4.26]) implies that this decomposition as a direct sum is unique up to an isomorphism. Its proof follows the same lines as [16, Proposition 1.2] for modules. We now need the following lemma:

Lemma 7 *Let $\hat{\mathcal{P}}_n = \oplus_{k=0}^{n} \mathbb{C}_k[y_0,y_1,y_2,y_3]$. We consider the following representation:*

$$\begin{array}{rcl} \hat{\varphi}: SO(1,3)_e & \longrightarrow & GL(\hat{\mathcal{P}}_n) \\ L & \longmapsto & \{R(y_0,y_1,y_2,y_3) \longmapsto R(L^{-1}(y_0,y_1,y_2,y_3))\} \end{array}$$

Denote by $(\hat{\Phi}, \hat{\mathcal{P}}_n)$ the associated representation of $so_{\mathbb{C}}(1,3)$. A non-zero polynomial $Q \in \hat{\mathcal{P}}_n$ is said to be a highest weight vector associated to the weight $(n-2j,0)$ of the representation $(\hat{\Phi}, \hat{\mathcal{P}}_n)$ of $so_{\mathbb{C}}(1,3)$ if

$$\hat{\Phi}(D_1)Q = (n-2j)Q, \qquad \hat{\Phi}(D_2)Q = 0, \qquad \hat{\Phi}(C_1)Q = 0, \qquad \hat{\Phi}(C_3)Q = 0, \tag{58}$$

with $D_1 = iR_3$, $D_2 = L_3$, $C_1 = R_1 + L_2 + i(R_2 + L_1)$ and $C_3 = R_1 - L_2 + i(R_2 - L_1)$, where the matrices R_j and L_j are defined by (53) and (54).

A polynomial $Q \in \hat{\mathcal{P}}_n$ satisfies (58) if and only if

$$Q(y_0,y_1,y_2,y_3) = (y_1 - iy_2)^{n-2j} \sum_{k=0}^{j} \lambda_k (y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k} \tag{59}$$

with $\lambda_k \in \mathbb{C}$ for $k = 0, \ldots, j$.

Proof. Let us assume that $Q \in \hat{\mathcal{P}}_n$ satisfies (58). As previously, by [10, Proposition 4.4] and [10, Proposition 4.6], we have

$$\hat{\Phi}(Z) = \frac{d}{dt}\tilde{\varphi}(e^{tZ_1})\Big|_{t=0} + i\frac{d}{dt}\tilde{\varphi}(e^{tZ_2})\Big|_{t=0}, \qquad Z = Z_1 + iZ_2, \qquad Z_1, Z_2 \in so_{\mathbb{R}}(1,3).$$

The change of variables $Y = PX$ where

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad X = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

diagonalizes both D_1 and D_2 and implies that the coefficients of

$$\tilde{Q}(x_0, x_1, x_2, x_3) = \sum_{k_0+k_1+k_2+k_3 \leq n} a_{k_0,k_1,k_2,k_3}\, x_0^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3},$$

where $\tilde{Q}(X) = Q(PX)$, satisfy

$$\begin{aligned} (n - 2j - k_1 + k_2) a_{k_0,k_1,k_2,k_3} &= 0, && (60) \\ (k_3 - k_0) a_{k_0,k_1,k_2,k_3} &= 0, && (61) \\ (k_2+1) a_{k_0-1,k_1,k_2+1,k_3} + (k_3+1) a_{k_0,k_1-1,k_2,k_3+1} &= 0, \qquad k_0 \geq 1, k_1 \geq 1, && (62) \\ (k_0+1) a_{k_0+1,k_1-1,k_2,k_3} + (k_2+1) a_{k_0,k_1,k_2+1,k_3-1} &= 0, \qquad k_1 \geq 1, k_3 \geq 1. && (63) \end{aligned}$$

for every $(k_0, k_1, k_2, k_3) \in \mathbb{N}^4$, $k_0 + k_1 + k_2 + k_3 \leq n$. We thus deduce that

$$\tilde{Q}(x_0, x_1, x_2, x_3) = \sum_{l=0}^{j} \sum_{m=0}^{j-l} a_{m,n-j-m-l,j-m-l,m}\, x_0^{m}\, x_1^{n-j-m-l}\, x_2^{j-m-l}\, x_3^{m}.$$

Let $l_0 \in [\![0, j]\!]$ and $m_0 \in [\![0, j-l]\!]$ be such that $a_{m_0,n-j-m_0-l_0,j-m_0-l_0,m_0} \neq 0$. Equations (62) and (63) imply that

$$a_{m,n-j-m-l,j-m-l,m} = (-1)^{j-m-l} \binom{j-l}{m} \frac{a_{m_0,n-j-m_0-l_0,j-m_0-l_0,m_0} (-1)^{j-m_0-l_0}}{\binom{j-l_0}{m_0}},$$

for each $l \in [\![0, j]\!]$ and $m \in [\![0, j-l]\!]$. Consequently,

$$\tilde{Q}(x_0, x_1, x_2, x_3) = x_1^{n-2j} \sum_{l=0}^{j} \lambda_l (x_0 x_3 - x_1 x_2)^{j-l},$$

with $\lambda_l \in \mathbb{C}$ for $l = 0, \ldots, j$. □

Proof of Theorem 1. Let W be an irreducible subrepresentation of $(\varphi, \mathcal{P}_n)$.

Let us recall that, by [10, Proposition 4.33], for every finite dimensional representation (Π, V) of a Lie group that decomposes as a direct sum of irreducible representations, every stable subspace of V also decomposes as a direct sum of irreducible representations and, given a stable subspace U of V, there is a stable subspace $\tilde{U}$ such that $V = U \oplus \tilde{U}$.

Case 1: There exists $k \in [\![0, n]\!]$ such that $W \subset \mathbb{R}_k[y_0, y_1, y_2, y_3]$.

It follows from (55) and [10, Proposition 4.33] that there exists a stable subspace $W' \subset \mathbb{R}_k[y_0, y_1, y_2, y_3]$ such that $\mathbb{R}_k[y_0, y_1, y_2, y_3] = W \oplus W'$. Then, [10, Proposition 4.33] implies that $W' = \oplus_\alpha \Gamma_\alpha$ and, thus,

$$\mathbb{R}_k[y_0, y_1, y_2, y_3] = W \bigoplus \oplus_\alpha \Gamma_\alpha.$$

By uniqueness of (55), there exists $j \in [\![0, [k/2]]\!]$ such that $W \simeq \tilde{\Gamma}^{(k)}_{k-2j,0}$. Considering the complexification $W_{\mathbb{C}} = W + iW$ of W and extending the action of $SO(1,3)_e$ on $W_{\mathbb{C}}$ to an action of $so_{\mathbb{C}}(1,3)$, we deduce from [10, Proposition 7.15] that $W_{\mathbb{C}}$ contains a unique highest weight vector Q_{k-2j} associated to the weight $(k-2j, 0)$. Lemma 7 implies the existence of $\lambda \in \mathbb{C}$ such that

$$Q_{k-2j}(y_0, y_1, y_2, y_3) = \lambda (y_1 - iy_2)^{k-2j} (y_0^2 - y_1^2 - y_2^2 - y_3^2)^j.$$

By [10, Proposition 7.18], since $W_{\mathbb{C}}$ is a complex irreducible representation, $W_{\mathbb{C}}$ is generated by the iterated action of $\hat{\Phi}(C_2)$ and $\hat{\Phi}(C_4)$ on Q_{k-2j}, with $C_2 = R_2 + L_1 + i(R_1 + L_2)$ and $C_4 = R_2 - L_1 + i(R_1 - L_2)$. Consequently, $W_{\mathbb{C}} = \Gamma^{(k)}_{k-2j,0}$ and $W = \tilde{\Gamma}^{(k)}_{k-2j,0}$.

Case 2: There is no k such that W is included in $\mathbb{R}_k[y_0, y_1, y_2, y_3]$.

By (57) and uniqueness of this decomposition, we deduce, as previously, that there exists $j \in [\![0, [n/2]]\!]$ such that $W \simeq \Gamma_{n-2j,0}$. Considering the complexification $W_{\mathbb{C}} = W + iW$ of W and extending the action of $SO(1,3)_e$ on $W_{\mathbb{C}}$ to an action of $so_{\mathbb{C}}(1,3)$, we deduce that $W_{\mathbb{C}}$ contains a highest weight vector Q_{n-2j} associated to the weight $(n-2j, 0)$. Lemma 7 implies the existence of constants $\lambda_k \in \mathbb{C}$ for $k = 0, \dots, j$ such that

$$Q_{n-2j}(y_0, y_1, y_2, y_3) = (y_1 - iy_2)^{n-2j} \sum_{k=0}^{j} \lambda_k (y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k}.$$

As previously, $W_{\mathbb{C}}$ is generated by the iterated action of $\hat{\Phi}(C_2)$ and $\hat{\Phi}(C_4)$ on Q_{n-2j}. Consequently, $W_{\mathbb{C}}$ is the complex vector space generated by

$$\left(\sum_{k=0}^{j} \lambda_k (y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k} \sum_{r=\max(l-m,0)}^{\min(l,n-2j-m)} \frac{(n-2j-m)!}{(n-2j-m-r)!} \frac{m!}{(m-l+r)!} \binom{l}{r} \right.$$
$$\left. (y_1 - iy_2)^{n-2j-m-r} (y_0 + y_3)^r (y_0 - y_3)^{m-l+r} (y_1 + iy_2)^{l-r} \right)_{0 \le l,\, m \le n-2j}. \quad (64)$$

For $l = 0$ and $m = n - 2j$, we deduce that

$$\sum_{k=0}^{j} \lambda_k (y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k} (y_0 - y_3)^{n-2j}$$

belongs to $W_{\mathbb{C}}$. The coefficients λ_k may not be all equal to 0. Without loss of generality, we may assume that there exists $k \in \mathbb{N}$ such that $\Re(\lambda_k)$ is non-zero. (If not, it suffices to replace λ_k with $i\lambda_k$). We obtain that

$$\sum_{k=0}^{j} \Re(\lambda_k)(y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k}(y_0 - y_3)^{n-2j} \tag{65}$$

is non-zero. By definition of $W_{\mathbb{C}}$, the polynomial (65) belongs to $W_{\mathbb{C}}$. Applying $\hat{\Phi}(C_1)$ $n-2j$ times to (65) leads to

$$\sum_{k=0}^{j} \Re(\lambda_k)(y_0^2 - y_1^2 - y_2^2 - y_3^2)^{j-k}(y_1 - iy_2)^{n-2j}. \tag{66}$$

But $W_{\mathbb{C}}$ is stable under $\hat{\Phi}(C_1)$, thus the polynomial (66) belongs to $W_{\mathbb{C}}$. By Lemma 7, the polynomial (66) is a highest weight vector associated to the weight $(n-2j, 0)$. By [10, Proposition 7.15], $W_{\mathbb{C}}$ contains, up to a constant, a unique highest weight vector associated to the weight $(n-2j, 0)$. Consequently, the coefficients λ_k of (64) are real. Finally, we deduce that W is the real vector space generated by the real parts and the imaginary parts of (20) with $\lambda_k \in \mathbb{R}$ for $k = 0, \dots, j$. $\square$

A Representation theory in the classical case

Thanks to the representation theory of Lie groups and Lie algebras, we can determine the finite dimensional subspace of $\mathbb{R}[v_1, v_2, v_3]$ that are stable under any rotation of $SO(3)$. We consider the following action of $SO(3)$ on the subspace $\mathcal{P}_n$ composed of the polynomials of $\mathbb{R}[y_1, y_2, y_3]$ with total degree less or equal to n.

$$\begin{array}{rcl} \varphi : SO(3) & \longrightarrow & GL(\mathcal{P}_n) \\ L & \longmapsto & \{R(y_1, y_2, y_3) \mapsto R(L^{-1}(y_1, y_2, y_3))\} \end{array} \tag{67}$$

We first consider the restriction of φ to $\mathbb{R}_n[y_1, y_2, y_3]$, the set of real homogeneous polynomials with degree n. The representation theory of Lie groups and Lie algebras then implies that

Theorem 8 *The representation $(\varphi_{|\mathbb{R}_n[y_1,y_2,y_3]}, \mathbb{R}_n[y_1, y_2, y_3])$ of $SO(3)$ is not irreducible. We have the following decomposition:*

$$\mathbb{R}_n[y_1, y_2, y_3] = \bigoplus_{j=0}^{[n/2]} \overline{\Gamma}_{2n-4j}^{(n)}, \tag{68}$$

where the space $\overline{\Gamma}^{(n)}_{2n-4j}$ *is generated by the real parts and the imaginary parts of*

$$(y_1^2+y_2^2+y_3^2)^j \sum_{m=[(l+1)/2]}^{\min(l,n-2j)} \frac{(-1)^{m+l}\,(n-2j)!\,l!}{(n-2j-m)!(l-m)!(2m-l)!\,2^{n-2j+l-2m}} \\ y_1^{2m-l}(y_2-iy_3)^{n-2j-m}(y_2+iy_3)^{l-m}, \quad (69)$$

for $l \in [\![0, 2n-4j]\!]$. *The subrepresentations* $(\varphi_{|\overline{\Gamma}^{(n)}_{2n-4j}}, \overline{\Gamma}^{(n)}_{2n-4j})$ *are irreducible.*

The proof of Theorem 8 follows the same lines as the proof of Theorem 6. As we deduced Theorem 1 from Theorem 6, we deduce the following theorem from Theorem 8.

Theorem 9 *A space* W *is an irreducible subrepresentation of* $(\varphi, \mathcal{P}_n)$ *if and only if there exist* $j \in [\![0, [n/2]]\!]$ *and some real numbers* $(\lambda_k)_{0\le k\le j}$ *such that* W *is generated by the real parts and the imaginary parts of*

$$\sum_{k=0}^{j} \lambda_k (y_1^2+y_2^2+y_3^2)^k \sum_{m=[(l+1)/2]}^{\min(l,n-2j)} \frac{(-1)^{m+l}\,(n-2j)!\,l!}{(n-2j-m)!(l-m)!(2m-l)!\,2^{n-2j+l-2m}} \\ y_1^{2m-l}(y_2-iy_3)^{n-2j-m}(y_2+iy_3)^{l-m}. \quad (70)$$

for $l \in [\![0, 2n-4j]\!]$.

This theorem describes all the irreducible representations of $(\varphi, \mathcal{P}_n)$. We then obtain all the finite dimensional subspaces of $\mathbb{R}[v_1, v_2, v_3]$ that are stable under any rotation. We have the following proposition.

Proposition 10 *For every* $r \in \mathbb{N}$, $j \in [\![0, [r/2]]\!]$, *let* $\mathbb{T}_{r,j}$ *denote the vector space generated by the real parts and the imaginary parts of*

$$\sum_{k=0}^{j} \lambda_k |v|^{2k} \sum_{m=[(l+1)/2]}^{\min(l,r-2j)} \frac{(-1)^{m+l}\,(r-2j)!\,l!}{(r-2j-m)!(l-m)!(2m-l)!\,2^{r-2j+l-2m}} \\ v_1^{2m-l}(v_2-iv_3)^{r-2j-m}(v_2+iv_3)^{l-m}.$$

for $l \in [\![0, 2r-4j]\!]$. *Each* $\mathbb{T}_{r,j}$ *is stable under any rotation.*

Moreover, a finite dimensional subspace $\mathbb{T}$ *of* $\mathbb{R}[v_1, v_2, v_3]$ *is stable under any rotation if and only if there exist* $N \in \mathbb{N}$ *and some* $r_k \in \mathbb{N}$ *and* $j_k \in [\![0, [r_k/2]]\!]$, $k = 1, \ldots, N$ *such that* $\mathbb{T}$ *is the vector sum of the* $\mathbb{T}_{r_k,j_k}$, $k = 1, \ldots, N$.

Let us now show that, for each $r \in \mathbb{N}$, the spaces $\mathbb{T}_{r,0}$ are generated by the components of some tensors.

Theorem 11 *Let $l \in \mathbb{N}$. For any tensor T of order l, we denote by $\overline{T}$ the symmetric part of T, that is the tensor whose components are*

$$\overline{T}^{j_1,\dots,j_l} = \frac{1}{l!} \sum_{\sigma \in \Sigma_l} T^{j_{\sigma(1)},\dots,j_{\sigma(l)}}, \qquad (j_1,\dots,j_l) \in [\![1,3]\!]^l$$

where Σ_l denotes the symmetric group of order l.

Then, the vector space $\mathbb{T}_{r,0}$ given by Proposition 10 is generated by the components of the tensor $S_r(v)$ defined by

$$S_r(v) = \mathcal{T}_r(v) + \sum_{k=1}^{[r/2]} \frac{(-1)^k \, r! \, (r-1)! \, (2r-2k)!}{2\,(r-2k)! \, k! \, (r-k)! \, (2r-1)!} \, |v|^{2k} \, \overline{\underbrace{I_3 \otimes \ldots \otimes I_3}_{k \text{ times}} \otimes \mathcal{T}_{r-2k}(v)}, \quad (71)$$

where $\mathcal{T}_r(v) = \otimes^r v$ and I_3 is the identity matrix of order 3.

We now write down the moment spaces that arise in (68) for $n = 2, 3, 4$. Since we look here for moment spaces that are compatible with the Galilean invariance, we also consider the stability under the translations. Moreover, we are only interested in moment spaces that generalize the fluid dynamic approximation and thus contain the mass 1, the velocity v and the energy $|v|^2$.

Case $n = 2$

By Theorem 8, we have

$$\mathbb{R}_2[v_1, v_2, v_3] = \overline{\Gamma}_4^{(2)} \oplus \overline{\Gamma}_0^{(2)},$$

with $\overline{\Gamma}_0^{(2)} = \mathrm{span}\,(|v|^2)$ and

$$\overline{\Gamma}_4^{(2)} = \mathrm{span}\,((v_i v_j)_{i \neq j},\, v_1^2 - v_3^2,\, v_2^2 - v_3^2).$$

We add the mass and the velocity to $\overline{\Gamma}_0^{(2)}$ and obtain the moment space $\mathrm{span}\,(1, v, |v|^2)$.

The space $\overline{\Gamma}_4^{(2)}$ is a 5-dimensional space. Adding 1, v and $|v|^2$, we obtain the 10-dimensional space $\mathrm{span}\,(1,\, v,\, v \otimes v)$ which is stable under any rotation and any translation.

Case $n = 3$

We infer from (68) that

$$\mathbb{R}_3[v_1, v_2, v_3] = \overline{\Gamma}_6^{(3)} \oplus \overline{\Gamma}_2^{(3)},$$

where $\overline{\Gamma}_2^{(3)} = \mathrm{span}\,(v|v|^2)$ and

$$\begin{aligned} \overline{\Gamma}_6^{(3)} \;=\; & \mathrm{span}\,(v_1 v_2 v_3,\, v_1(v_1^2 - 3v_2^2),\, v_1(v_2^2 - v_3^2),\, v_2(v_1^2 - v_3^2), \\ & v_2(v_2^2 - 3v_3^2),\, v_3(v_1^2 - v_2^2),\, v_3(3v_2^2 - v_3^2)). \end{aligned}$$

The space $\overline{\Gamma}_2^{(3)}$ is a 3-dimensional space. Adding 1, v and $|v|^2$, we obtain the 8-dimensional space span $(1,\, v,\, |v|^2,\, v|v|^2)$ which is stable under any rotation but not under the translations. In order to make it stable under any translation, we add $v \otimes v$ and obtain the Grad 13-moment system

$$\text{span}\,(1,\, v,\, v \otimes v,\, v|v|^2)$$

The space $\overline{\Gamma}_6^{(3)}$ is a 7-dimensional space. Adding 1, v and $|v|^2$, we obtain the 12-dimensional space

$$\begin{aligned}&\text{span}\,(1,\, v,\, |v|^2,\, v_1v_2v_3,\, v_1(v_1^2-3v_2^2),\, v_1(v_2^2-v_3^2),\\ &\qquad v_2(v_1^2-v_3^2),\, v_2(v_2^2-3v_3^2),\, v_3(v_1^2-v_2^2),\, v_3(3v_2^2-v_3^2)).\end{aligned}$$

This space is not stable under the translations. Consequently, we add $v \otimes v$ and get

$$\begin{aligned}&\text{span}\,(1,\, v,\, v\otimes v,\, v_1v_2v_3,\, v_1(v_1^2-3v_2^2),\, v_1(v_2^2-v_3^2),\\ &\qquad v_2(v_1^2-v_3^2),\, v_2(v_2^2-3v_3^2),\, v_3(v_1^2-v_2^2),\, v_3(3v_2^2-v_3^2)).\end{aligned}$$

which is stable under any rotation and under any translations. This space has dimension 17. We can also add $v \otimes v \otimes v$ and obtain the system

$$\text{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v),$$

which has dimension 20.

Case $n = 4$
By (68), we have

$$\mathbb{R}_4[v_1, v_2, v_3] = \overline{\Gamma}_8^{(4)} \oplus \overline{\Gamma}_4^{(4)} \oplus \overline{\Gamma}_0^{(4)},$$

where $\overline{\Gamma}_0^{(4)} = \text{span}\,(|v|^4)$,

$$\overline{\Gamma}_4^{(4)} = |v|^2\,\text{span}\,((v_iv_j)_{i\neq j},\, v_1^2-v_3^2,\, v_2^2-v_3^2),$$

and

$$\begin{aligned}\overline{\Gamma}_8^{(4)} \;=\; &\text{span}\,(v_2^4-6v_2^2v_3^2+v_3^4,\, 8v_1^4-24v_1^2(v_2^2+v_3^2)+3(v_2^2+v_3^2)^2,\, v_2v_3(v_2^2-v_3^2),\\ &v_1v_2(v_2^2-3v_3^2),\, v_1v_3(3v_2^2-v_3^2),\, v_1v_2(4v_1^2-3v_2^2-3v_3^2),\\ &v_1v_3(4v_1^2-3v_2^2-3v_3^2),\, v_2v_3(6v_1^2-v_2^2-v_3^2),\, v_3^4-6v_1^2v_3^2+6v_1^2v_2^2-v_2^4).\end{aligned}$$

The space $\overline{\Gamma}_0^{(4)}$ is a 1-dimensional space. Adding 1, v and $|v|^2$, we obtain the 6-dimensional space span $(1,\, v,\, |v|^2,\, |v|^4)$ which is stable under any rotation but not under the translations. In order to make it stable under any translation, we add $v \otimes v$ and $v|v|^2$. We then get

$$\text{span}\,(1,\, v,\, v \otimes v,\, v|v|^2, |v|^4),$$

which is a 14-dimensional space. We can also add $v \otimes v \otimes v$ instead of $v|v|^2$ and obtain

$$\mathrm{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v,\, |v|^4),$$

which is a 21-dimensional space.

The space $\overline{\Gamma}_4^{(4)}$ is a 5-dimensional space. Adding 1, v and $|v|^2$, we obtain the 10-dimensional space

$$\mathrm{span}\,(1,\, v,\, |v|^2,\, |v|^2(v_iv_j)_{i\neq j},\, |v|^2(v_1^2 - v_3^2),\, |v|^2(v_2^2 - v_3^2)).$$

This space is not stable under the translations. Consequently, we add $v\otimes v$ and $v\otimes v\otimes v$. We get

$$\mathrm{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v,\, |v|^2(v_iv_j)_{i\neq j},\, |v|^2(v_1^2 - v_3^2),\, |v|^2(v_2^2 - v_3^2)).$$

which is stable under any rotation and any translation. This space has dimension 25. We could also add either $|v|^2 v \otimes v$ or $v \otimes v \otimes v \otimes v$ to the previous space. We would then get

$$\mathrm{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v,\, |v|^2 v \otimes v),$$

and

$$\mathrm{span}\,(1,\, v,\, v \otimes v,\, v \otimes v \otimes v,\, v \otimes v \otimes v \otimes v).$$

We add 1, v and $|v|^2$ to the space $\overline{\Gamma}_8^{(4)}$ and obtain the space

$$\begin{aligned}\mathrm{span}\,(1, v, |v|^2, &v_2^4 - 6v_2^2v_3^2 + v_3^4,\, 8v_1^4 - 24v_1^2(v_2^2+v_3^2) + 3(v_2^2+v_3^2)^2,\\ &v_2v_3(v_2^2 - v_3^2),\, v_1v_2(v_2^2 - 3v_3^2),\, v_1v_3(3v_2^2 - v_3^2),\, v_1v_2(4v_1^2 - 3v_2^2 - 3v_3^2),\\ &v_1v_3(4v_1^2 - 3v_2^2 - 3v_3^2),\, v_2v_3(6v_1^2 - v_2^2 - v_3^2),\, v_3^4 - 6v_1^2v_3^2 + 6v_1^2v_2^2 - v_2^4).\end{aligned}$$

But this space is not stable under the translations and we need to add $v\otimes v$ and $v\otimes v\otimes v$. We get the following 29-dimensional space

$$\begin{aligned}\mathrm{span}\,(1, v, v\otimes v, v\otimes v\otimes v, &v_2^4 - 6v_2^2v_3^2 + v_3^4,\, 8v_1^4 - 24v_1^2(v_2^2+v_3^2) + 3(v_2^2+v_3^2)^2,\\ &v_2v_3(v_2^2 - v_3^2),\, v_1v_2(v_2^2 - 3v_3^2),\, v_1v_3(3v_2^2 - v_3^2),\, v_1v_2(4v_1^2 - 3v_2^2 - 3v_3^2),\\ &v_1v_3(4v_1^2 - 3v_2^2 - 3v_3^2),\, v_2v_3(6v_1^2 - v_2^2 - v_3^2),\, v_3^4 - 6v_1^2v_3^2 + 6v_1^2v_2^2 - v_2^4).\end{aligned}$$

Conclusion
The classical moment spaces with maximal degree 2, 3 or 4 are

degree = 2	$\mathrm{span}(1, v, \lvert v\rvert^2)$,	(admissible)
	$\mathrm{span}(1, v, v\otimes v)$,	(admissible)
degree = 3	$\mathrm{span}(1, v, v\otimes v, v\lvert v\rvert^2)$,	(non admissible)
	$\mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v - 3/5\lvert v\rvert^2\overline{I_3\otimes v})$,	(non admissible)
	$\mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v)$,	(non admissible)

$$
\begin{array}{lll}
\text{degree} = 4 & \mathrm{span}(1, v, v\otimes v, v|v|^2, |v|^4), & \text{(admissible)} \\
& \mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v, |v|^4), & \text{(admissible)} \\
& \mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v, |v|^2(v\otimes v - |v|^2/3I_3)), & \text{(non admissible)} \\
& \mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v, |v|^2 v\otimes v), & \text{(admissible)} \\
& \mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v & \\
& \qquad - 6|v|^2/7\overline{I_3\otimes v\otimes v} + 3|v|^4/35 I_3\otimes I_3), & \text{(non admissible)} \\
& \mathrm{span}(1, v, v\otimes v, v\otimes v\otimes v, v\otimes v\otimes v\otimes v), & \text{(admissible)}
\end{array}
$$

which have respectively dimension 5, 10, 13, 17, 20, 14, 21, 25, 26, 29 and 35.

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