Convergence of a discrete version of the Oort-Hulst-Safronov coagulation equation

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Abstract

A discrete version of the Oort-Hulst-Safronov (OHS) coagulation equation is studied. Besides the existence of a solution to the Cauchy problem, it is shown that solutions to a suitable sequence of those discrete equations converge towards a solution to the OHS equation.

1 Introduction

We study here a discrete approximation to the Oort-Hulst-Safronov (OHS) coagulation equation which describes the growth by accretion of stellar objects. More generally, coagulation models aim at describing the process by which particles encounter and merge into a single one, each particle being fully identified by a size variable (volume, mass, length, ...). The evolution of these particles is then described by a size distribution function $f(t,x) \geq 0$ which represents the density of particles of size $x \in (0,+\infty)$ at time $t \geq 0$. Besides the classical Smoluchowski coagulation equation originally introduced in colloidal chemistry (see [1, 2] and the references therein), a different coagulation equation has been derived independently in an astrophysical context by Oort and van de Hulst [3] and reads [4]

$$\partial_t f = Q(f), \qquad (t,x) \in (0,+\infty) \times \mathbb{R}_+, \tag{1}$$

$$f(0,x) = f^{in}(x), \qquad x \in \mathbb{R}_+, \tag{2}$$

where

$$Q(f) = -\partial_x \left[f(t,x) \int_0^x y\, K(x,y) f(t,y)\, dy \right] - \int_x^\infty K(x,y) f(t,x) f(t,y)\, dy. \tag{3}$$

Here, ∂_t and ∂_x denote the partial derivatives with respect to time t and size x, respectively, and the coagulation kernel K is a non-negative and symmetric function accounting for the physics of the coalescence process. Approximating ∂_x by an upwind finite difference scheme and the integrals by Riemann sums, one obtains a discrete version of (1) which reads

$$\frac{dc_i}{dt} = Q_i(c) \qquad \text{in} \quad (0,+\infty), \tag{4}$$

$$c_i(0) = c_i^{in}, \tag{5}$$

for $i \geq 1$, where $c = (c_i)_{i\geq 1}$,

$$Q_i(c) = c_{i-1} \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j - c_i \sum_{j=1}^{i} j\, K_{i,j}\, c_j - \sum_{j=i}^{\infty} K_{i,j}\, c_i\, c_j, \tag{6}$$

and $K_{i,j} = K_{j,i} \geq 0$ is the discrete coagulation kernel. Equations (4)-(6) are also a particular case of a two-parameter family of discrete coagulation models introduced by Dubovski [5], where a link with the OHS equation is highlighted. In the sequel, equations (4)-(6) will be referred to as the discrete OHS (dOHS) equation.

Our aim is here to justify the connection between the OHS and dOHS equations. A similar relationship has been established in [6] between the classical continuous and discrete coagulation-fragmentation equations. We adapt herein the approach developed there. Roughly speaking, the main idea is to realize that the dOHS equation may be seen as a modified OHS equation. More precisely, let $c = (c_i)_{i\geq 1}$ be a solution to (4)-(6) and $(\varphi_i)_{i\geq 1}$ a sequence of (sufficiently rapidly decaying) real numbers. Then, thanks to the symmetry property $K_{i,j} = K_{j,i}$, a weak formulation for (4)-(6) reads

$$\frac{d}{dt} \sum_{i=1}^{\infty} c_i\, \varphi_i = \sum_{i=1}^{\infty} \sum_{j=1}^{i} K_{i,j}\, c_i\, c_j \left[j\, (\varphi_{i+1} - \varphi_i) - \varphi_j \right]. \tag{7}$$

We show that (7) may be interpreted as a weak formulation of a modified OHS equation. To this end, we fix $\varepsilon \in (0,1)$ and set

$$\Lambda_i^\varepsilon = [(i-1/2)\varepsilon, (i+1/2)\varepsilon) \qquad \text{and} \qquad \chi_i^\varepsilon = \mathbf{1}_{\Lambda_i^\varepsilon}, \tag{8}$$

for $i \geq 1$. We next introduce, for $(t,x,y) \in \mathbb{R}_+^3$,

$$f_\varepsilon(t,x) = \sum_{i=1}^{\infty} c_i(t)\, \chi_i^\varepsilon(x) \qquad \text{and} \qquad K_\varepsilon(x,y) = \sum_{i,j=1}^{\infty} \frac{K_{i,j}}{\varepsilon}\, \chi_i^\varepsilon(x)\, \chi_j^\varepsilon(y).$$

For $\varphi \in \mathcal{D}(\mathbb{R}_+)$, we define the approximated ε-step function of φ by

$$\varphi_\varepsilon(x) = \sum_{i=1}^{\infty} \varphi_i^\varepsilon\, \chi_i^\varepsilon(x) \qquad \text{with} \qquad \varphi_i^\varepsilon = \frac{1}{\varepsilon} \int_{\Lambda_i^\varepsilon} \varphi(y)\, dy, \tag{9}$$

for every $x \in \mathbb{R}_+$. Finally, for any function g from $\mathbb{R}_+$ to $\mathbb{R}$ of the form

$$g(x) = \sum_{i=1}^{\infty} g_i\, \chi_i^\varepsilon(x), \qquad g_i \in \mathbb{R},$$

we define the discrete size derivative $D_\varepsilon(g)$ of g by

$$D_\varepsilon(g)(x) = \frac{1}{\varepsilon}\sum_{i=1}^{\infty}(g_{i+1}-g_i)\,\chi_i^\varepsilon(x), \qquad x\in\mathbb{R}_+.$$

As we shall see in Lemma 11 below, for $\varphi\in\mathcal{D}(\mathbb{R}_+)$, $(\varphi_\varepsilon, D_\varepsilon(\varphi_\varepsilon))$ converges a.e. towards $(\varphi,\partial_x\varphi)$, the function φ_ε being defined by (9).

With the previous notations, (7) reads

$$\frac{d}{dt}\left(\int_0^\infty f_\varepsilon\,\varphi_\varepsilon\,dx\right) = \int_0^\infty\int_0^{r_\varepsilon(x)} K_\varepsilon(x,y)\,f_\varepsilon(t,x)\,f_\varepsilon(t,y)\,[\,y\,D_\varepsilon(\varphi_\varepsilon)(x)-\varphi_\varepsilon(y)]\,dy\,dx, \quad (10)$$

for $\varphi\in\mathcal{D}(\mathbb{R}_+)$, where

$$r_\varepsilon(x) = \left(\left[\frac{x}{\varepsilon}+\frac{1}{2}\right]+\frac{1}{2}\right)\varepsilon, \quad (11)$$

denoting by $[u]$ the integer part of the real number u.

Thus, if we suppose that (f_ε) converges towards some function f and if (K_ε) converges towards some K, then we may pass formally to the limit in (10). Thereby, we obtain that f satisfies, for every $\varphi\in\mathcal{D}(\mathbb{R}_+)$,

$$\frac{d}{dt}\int_0^\infty f\,\varphi\,dx = \int_0^\infty\int_0^x K(x,y)f(t,x)f(t,y)\,[\,y\,\partial_x\varphi(x)-\varphi(y)]\,dy\,dx,$$

which turns out to be the weak formulation of the OHS equation (1) (see (17) below). Observe that the convergence of K_ε to a finite limit requires that $K_{i,j}$ depends on ε.

We now describe the contents of the paper. We first introduce a sequence of approximated discrete equations of the OHS equation and state our main results in the next section. We then show, in Section 3, the convergence of a sequence of solutions to these discrete models towards a solution to the OHS model. For the sake of completeness, the Cauchy problem for the dOHS equation is investigated in Section 4. We finally illustrate the convergence theorem by a numerical comparison between an explicit solution and the associated discrete solution.

2 Main results

We make here the same assumptions as in [7]: we require that the classical symmetry condition is fulfilled, namely

$$0\le K(x,y)=K(y,x), \qquad (x,y)\in\mathbb{R}_+^2, \quad (12)$$

and that

$$K\in W^{1,\infty}_{loc}([0,+\infty)^2), \quad (13)$$

$$\partial_x K(x,y)\ge -\alpha, \qquad \text{for some } \alpha\ge 0. \quad (14)$$

We also suppose that K is *strictly subquadratic*, that is, for each $R \geq 1$,

$$\omega_R(y) = \sup_{x\in[0,R]} \frac{K(x,y)}{y} \longrightarrow 0 \quad \text{as} \quad y \to +\infty. \tag{15}$$

Concerning the initial condition, we assume that

$$f^{in} \in L^1_1(\mathbb{R}_+) = L^1(\mathbb{R}_+,(1+x)dx) \qquad \text{and} \qquad f^{in} \geq 0 \quad a.e. \tag{16}$$

The notion of weak solutions to the OHS equation we consider here is the same as in [7] and is as follows:

Definition 1 *Assume that K satisfies (12)-(15) and that f^{in} satisfies (16). A function $f = f(t,x)$ is said to be a weak solution to the OHS equation (1)-(3) with initial condition f^{in} if*

$$0 \ \leq \ f \in \mathcal{C}([0,T]; w - L^1(\mathbb{R}_+)) \ \cap \ L^\infty(0,T; L^1_1(\mathbb{R}_+)) \qquad \textit{for every } T \in \mathbb{R}_+,$$

and, for all $\varphi \in \mathcal{D}(\mathbb{R}_+)$ and $t > 0$,

$$\begin{aligned}\int_0^\infty f(t,x)\,\varphi(x)\,dx - \int_0^\infty f^{in}(x)\,\varphi(x)\,dx \\ = \int_0^t \int_0^\infty \int_0^x K(x,y)\,f(s,x)\,f(s,y)\,[\,y\,\partial_x\varphi(x) - \varphi(y)]\,dy\,dx\,ds.\end{aligned} \tag{17}$$

Here $\mathcal{C}([0,T]; w - L^1(\mathbb{R}_+))$ denotes the space of weakly continuous functions in $L^1(\mathbb{R}_+)$, that is the space of continuous functions from $[0,T]$ in $L^1(\mathbb{R}_+)$ endowed with its weak topology. Recall that it follows from [7, Theorem 2.2] that there exists at least a weak solution to the OHS equation (1)-(3) in the sense of Definition 1 when K and f^{in} fulfill (12)-(15) and (16), respectively.

We now introduce the discrete approximations to the OHS equation. We fix $\varepsilon \in (0,1)$ and define discrete coefficients $K^\varepsilon_{i,j}$ either by

$$K^\varepsilon_{i,j} = \frac{1}{\varepsilon} \int_{\Lambda^\varepsilon_i \times \Lambda^\varepsilon_j} K(x,y)\,dy\,dx, \tag{18}$$

or by

$$K^\varepsilon_{i,j} = \varepsilon K(\varepsilon i, \varepsilon j), \tag{19}$$

for $i,j \geq 1$. In both case, (12) and (15) imply that the following properties hold:

$$0 \quad \leq \quad K^\varepsilon_{i,j} \quad = \quad K^\varepsilon_{j,i}, \qquad i,j \geq 1 \tag{20}$$

$$\lim_{j\to\infty} \frac{K^\varepsilon_{i,j}}{j} \quad = \quad 0 \qquad \text{for each } i \geq 1. \tag{21}$$

We next define the discrete initial condition $c^{in,\varepsilon} = (c^{in,\varepsilon}_i)_{i\geq 1}$ by

$$c^{in,\varepsilon}_i = \frac{1}{\varepsilon} \int_{\Lambda^\varepsilon_i} f^{in}(x)\,dx, \qquad i \geq 1. \tag{22}$$

It is straightforward to check that

$$\varepsilon \sum_{i=1}^{\infty} c_i^{in,\varepsilon} \le \int_0^{\infty} f^{in}(x)\, dx, \tag{23}$$

and

$$\varepsilon^2 \sum_{i=1}^{\infty} i\, c_i^{in,\varepsilon} \le 2 \int_0^{\infty} x\, f^{in}(x)\, dx. \tag{24}$$

Let $c^\varepsilon = (c_i^\varepsilon)_{i\ge 1}$ be a solution to the dOHS equation (in the sense of Definition 3 below) with the coefficients $K_{i,j}^\varepsilon$ and the initial condition $c^{in,\varepsilon}$ such that

$$\sum_{i=1}^{\infty} i\, c_i^\varepsilon(t) \le \sum_{i=1}^{\infty} i\, c_i^{in,\varepsilon}, \qquad t \ge 0, \tag{25}$$

(see Section 4 for the existence of such a solution).

Similarly to what was done in Section 1, we introduce continuous formulations for the discrete quantities and set

$$f_\varepsilon(t,x) = \sum_{i=1}^{\infty} c_i^\varepsilon(t)\, \chi_i^\varepsilon(x), \tag{26}$$

$$K_\varepsilon(x,y) = \sum_{i,j=1}^{\infty} \frac{K_{i,j}^\varepsilon}{\varepsilon}\, \chi_i^\varepsilon(x)\, \chi_j^\varepsilon(y), \tag{27}$$

for $(t,x,y) \in \mathbb{R}_+^3$. With these notations, K_ε converges a.e. towards K and satisfies (12).

Our main results are the following.

Theorem 2 *Assume that K satisfies (12)-(15) and that f^{in} satisfies (16). We denote by c^ε a solution to the dOHS equation (4)-(6) with the coefficient $K_{i,j}^\varepsilon$ defined by (18) or (19) and with the initial data $c^{in,\varepsilon}$ defined by (22) such that (25) holds. Let f_ε be the function defined by (26). Then there exist a weak solution f to the OHS equation (1)-(3) with initial data f^{in} and a subsequence (f_{ε_n}) of (f_ε) such that*

$$f_{\varepsilon_n} \longrightarrow f \quad \text{in } \mathcal{C}([0,T]; w-L^1(\mathbb{R}_+)) \quad \text{for each } T \in \mathbb{R}_+.$$

As a by-product of Theorem 2, we obtain the existence of a weak solution to the OHS equation, thus providing, under the same set of assumptions, an alternative proof of [7, Theorem 2.2]. The existence proof in [7] also relies on weak compactness but with a different approximation scheme. Let us also point out that the approximation to the OHS equation developed in this paper might be used for numerical simulations, as illustrated in Section 5.

We also show the existence of solutions to the dOHS equation, arguing as in [8, 9]. Let us first give the definition of a weak solution to the dOHS equation.

Definition 3 *Let $T \in (0,+\infty)$ and assume that $c^{in} = (c_i^{in})_{i\geq 1}$ is a sequence of non-negative real numbers. A solution $c = (c_i)_{i\geq 1}$ to the dOHS equation (4)-(6) on $[0,T)$ is a sequence of non-negative continuous functions such that, for each $i \geq 1$ and $t \in (0,T)$,*

$$(i) \quad c_i \in \mathcal{C}([0,T)), \quad \sum_{j=i}^{\infty} K_{i,j}\, c_i\, c_j \in L^1(0,t),$$

$$(ii) \quad c_i(t) = c_i^{in} + \int_0^t \Big[c_{i-1} \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j - c_i \sum_{j=1}^{i} j\, K_{i,j}\, c_j - \sum_{j=i}^{\infty} K_{i,j}\, c_i\, c_j \Big]\, ds.$$

The existence result for the dOHS equation then reads:

Proposition 4 *Let $(K_{i,j})$ be a sequence of non-negative real numbers such that*

$$K_{i,j} = K_{j,i} \geq 0 \qquad \text{and} \qquad \lim_{k\to+\infty} \frac{K_{i,k}}{k} = 0, \qquad i,j \geq 1. \tag{28}$$

If $c^{in} = (c_i^{in})$ is a sequence of non-negative real numbers such that

$$\sum_{i=1}^{\infty} i\, c_i^{in} < +\infty, \tag{29}$$

then there exists at least a solution c to the dOHS equation (4)-(6) on $[0,+\infty)$ such that (25) holds.

We also note that, unlike the OHS equation, the dOHS equation propagates perturbations with an infinite speed. More precisely, it follows from [7, Theorem 2.6] that, if f^{in} is compactly supported in $[0,+\infty)$, then $f(t)$ is also compactly supported for $t \in [0,T_*)$, where T_* might be finite or infinite according to the growth of the coagulation kernel K. On the opposite, the following proposition holds for the dOHS equation.

Proposition 5 *Assume that $K_{i,i} > 0$ for $i \geq 1$. Let $c^{in} = (c_i^{in})_{i\geq 1}$ be a sequence of non-negative real numbers such that $c_k^{in} > 0$ for some $k \geq 1$ and $c = (c_i)_{i\geq 1}$ be a solution to the dOHS equation (4)-(6) on some interval $[0,T)$ with initial condition c^{in}. Then, for all $i \geq k$ and $t \in (0,T)$, $c_i(t) > 0$.*

3 Proof of Theorem 2

We consider here the dOHS equation (4)-(6) with the coefficient $K_{i,j}^{\varepsilon}$ defined by (18) or (19) and with the initial data $c^{in,\varepsilon}$ defined by (22). The proof is performed in two steps: the main idea relies on L^1 weak compactness. We thus need uniform estimates with respect to ε for the function f_ε defined by (26), which corresponds to the first step. These estimates ensure that (f_ε) lies in a weakly compact set of L^1. In a second step, we pass to the limit as $\varepsilon \to 0$.

3.1 *A priori* estimates

We set

$$M=\int_0^\infty f^{in}(x)\,(1+x)\,dx, \tag{30}$$

and notice that, by (26),

$$\int_0^\infty f_\varepsilon(t,x)\,dx=\varepsilon\sum_{i=1}^\infty c_i^\varepsilon(t) \qquad \text{and} \qquad \int_0^\infty f_\varepsilon(t,x)\,x\,dx=\varepsilon^2\sum_{i=1}^\infty i\,c_i^\varepsilon(t), \tag{31}$$

for every $t\geq 0$.

Lemma 6 *For all $t\geq 0$ and $\varepsilon\in(0,1)$, there holds*

$$\int_0^\infty f_\varepsilon(t,x)\,x\,dx\leq 2M. \tag{32}$$

Proof. Using successively (31), (25) and (24), we obtain that

$$\int_0^\infty f_\varepsilon(t,x)\,x\,dx=\varepsilon^2\sum_{i=1}^\infty i\,c_i^\varepsilon(t)\leq\varepsilon^2\sum_{i=1}^\infty i\,c_i^{in,\varepsilon}\leq 2\int_0^\infty f^{in}(x)\,x\,dx,$$

whence (32). □

Lemma 7 *For all $t\geq 0$ and $\varepsilon\in(0,1)$, we have*

$$\int_0^\infty f_\varepsilon(t,x)\,dx\leq M. \tag{33}$$

Proof. Let $m\geq 1$. Taking $\varphi_i=i$ for $i\leq m$ and $\varphi_i=0$ for $i>m$ in (7), we deduce from the non-negativity of $K_{i,j}^\varepsilon$ and c^ε that $\sum_{i=1}^m c_i^\varepsilon$ is a non-increasing function of time. Thus, for every $t\geq 0$,

$$\sum_{i=1}^m \varepsilon c_i^\varepsilon(t)\leq\sum_{i=1}^m \varepsilon c_i^{in,\varepsilon},$$

whence, by (23),

$$\sum_{i=1}^m \varepsilon c_i^\varepsilon(t)\leq\int_0^\infty f^{in}(x)\,dx. \tag{34}$$

We now let $m\to+\infty$ and deduce (33) thanks to (31). □

Lemma 8 *Let $\phi\in\mathcal{C}^2([0,+\infty))$ be a non-negative convex function such that $\phi(0)=0$, $\phi'(0)=1$ and ϕ' is concave. If*

$$L_\phi:=\int_0^\infty \phi(f^{in})(x)\,dx<+\infty, \tag{35}$$

then, for every $T\in\mathbb{R}_+$, there exists a constant $C(T)$ such that, for all $t\in[0,T]$ and $\varepsilon\in(0,1)$, we have

$$\int_0^\infty \phi(f_\varepsilon(t,x))\,dx\leq C(T)\,L_\phi. \tag{36}$$

Proof. The concavity of ϕ', the non-negativity of $\phi'(0)$ and $\phi(0)=0$ ensure that, for every $v \in \mathbb{R}_+$,

$$v\phi'(v) \le 2\,\phi(v), \tag{37}$$

(see [10, Lemma A.1] for a proof).

Let $T>0$, $R>0$ and $m \in \mathbb{N}$ such that $R \in \Lambda_m^\varepsilon$. We infer from (7) and the non-negativity of $K_{i,j}^\varepsilon$, c^ε and ϕ' that

$$\begin{aligned} \frac{d}{dt}\sum_{i=1}^m \phi(c_i^\varepsilon) \quad &\le \quad \sum_{i=1}^{m-1}\sum_{j=1}^{i} j\, c_i^\varepsilon\, c_j^\varepsilon\, K_{i,j}^\varepsilon\,[\phi'(c_{i+1}^\varepsilon)-\phi'(c_i^\varepsilon)] \\ &\quad - \sum_{j=1}^{m} j\, c_m^\varepsilon\, c_j^\varepsilon\, K_{m,j}^\varepsilon\, \phi'(c_m^\varepsilon). \end{aligned} \tag{38}$$

Introducing

$$\psi(x) := x\,\phi'(x) - \phi(x), \qquad x \in \mathbb{R}_+,$$

it easily follows from (37), the non-negativity and the convexity of ϕ that ψ satisfies the following properties:

$$0 \le \psi(x) \le x\,\phi'(x) \qquad \text{and} \qquad \psi(x) \le \phi(x), \qquad x \in \mathbb{R}_+. \tag{39}$$

Due to the convexity of ϕ, $\phi(x)-\phi(y) \ge (x-y)\phi'(y)$ for all $x,y\in\mathbb{R}_+$, and thus,

$$x(\phi'(y)-\phi'(x)) \le \psi(y)-\psi(x), \qquad x,y \in \mathbb{R}_+. \tag{40}$$

Using (39) and (40), we deduce from (38) that

$$\begin{aligned} \frac{d}{dt}\sum_{i=1}^m \phi(c_i^\varepsilon) \quad &\le \quad \sum_{i=1}^{m-1}\sum_{j=1}^{i} j\, c_j^\varepsilon\, K_{i,j}^\varepsilon\,[\,\psi(c_{i+1}^\varepsilon)-\psi(c_i^\varepsilon)\,] - \sum_{j=1}^{m} j\, c_j^\varepsilon\, K_{m,j}^\varepsilon\,\psi(c_m^\varepsilon) \\ &\le \quad \sum_{i=2}^{m}\sum_{j=1}^{i-1} j\, c_j^\varepsilon\,[\,K_{i-1,j}^\varepsilon - K_{i,j}^\varepsilon\,]\,\psi(c_i^\varepsilon). \end{aligned}$$

We infer from the definition of $K_{i,j}^\varepsilon$ by (18) or (19) and from (14) that

$$K_{i-1,j}^\varepsilon - K_{i,j}^\varepsilon \le \alpha\varepsilon^2 \qquad \text{for all } i \ge 2 \text{ and } j \ge 1.$$

Consequently, we obtain that

$$\frac{d}{dt}\sum_{i=1}^m \varepsilon\,\phi(c_i^\varepsilon) \le \alpha \left(\sum_{i=1}^m \varepsilon\,\psi(c_i^\varepsilon)\right)\left(\sum_{j=1}^\infty \varepsilon^2\, j\, c_j^\varepsilon\right).$$

By (39), (25) and (24), we have

$$\frac{d}{dt}\sum_{i=1}^m \varepsilon\,\phi(c_i^\varepsilon) \le 2\alpha M \sum_{i=1}^m \varepsilon\,\phi(c_i^\varepsilon).$$

Then, the successive use of the Gronwall Lemma and the Jensen inequality yields

$$\sum_{i=1}^{m} \varepsilon\,\phi(c_i^\varepsilon(t)) \le C(T) \sum_{i=1}^{m} \varepsilon\,\phi(c_i^{in,\varepsilon}) \le C(T) \sum_{i=1}^{m} \int_{\Lambda_i^\varepsilon} \phi(f^{in}(x))dx \le C(T) L_\phi,$$

for every $t \in [0,T]$. Finally, since $(m+1/2)\varepsilon > R$, we deduce that

$$\int_0^R \phi(f_\varepsilon(t,x))\,dx \le \varepsilon \sum_{i=1}^{m} \phi(c_i^\varepsilon(t)) \le C(T)\,L_\phi, \qquad t \in [0,T].$$

Letting now $R \to +\infty$ completes the proof of Lemma 8. □

Lemma 9 *For all* $T \in \mathbb{R}_+$ *and* $\psi \in \mathcal{C}_c^1([0,+\infty))$,

$$t \longmapsto \int_0^\infty f_\varepsilon(t,x)\,\psi(x)\,dx \quad \textit{is bounded in} \quad W^{1,\infty}(0,T). \tag{41}$$

Proof. Let $\psi \in \mathcal{C}_c^1([0,+\infty))$ such that $\mathrm{supp}(\psi) \subset [0,R]$ and set

$$\psi_i^\varepsilon = \frac{1}{\varepsilon} \int_{\Lambda_i^\varepsilon} \psi(x)\,dx, \qquad i \ge 1.$$

Denoting by m the integer such that $R \in \Lambda_m^\varepsilon$, we infer from (7) that

$$\begin{aligned} \left| \frac{d}{dt} \int_0^\infty f_\varepsilon(t,x)\,\psi(x)\,dx \right| &= \varepsilon \left| \frac{d}{dt} \sum_{i=1}^{\infty} c_i^\varepsilon\, \psi_i^\varepsilon \right| \\ &= \varepsilon \left| \sum_{i=1}^{m} \sum_{j=1}^{i} j\, c_i^\varepsilon\, c_j^\varepsilon\, K_{i,j}^\varepsilon\, (\psi_{i+1}^\varepsilon - \psi_i^\varepsilon) - \sum_{i=1}^{m} \sum_{j=i}^{\infty} \psi_i^\varepsilon\, K_{i,j}^\varepsilon\, c_i^\varepsilon\, c_j^\varepsilon \right|. \end{aligned}$$

Since

$$\left| \frac{\psi_{i+1}^\varepsilon - \psi_i^\varepsilon}{\varepsilon} \right| \le \|\psi\|_{W^{1,\infty}},$$

it follows from (24), (25) and (34) that

$$\left| \frac{d}{dt} \int_0^\infty f_\varepsilon(t,x)\,\psi(x)\,dx \right| \le M^2 \left[3\,\|K\|_{L^\infty((0,R+1)^2)} + 2 \sup_{j \ge m+1} \sup_{i \le m} \frac{K_{i,j}^\varepsilon}{j\varepsilon^2} \right] \|\psi\|_{W^{1,\infty}}.$$

By (15) and (27), there exists, for each $R \ge 1$, a bounded non-increasing function $\overline{\omega}_R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\sup_{y \ge M} \sup_{x \in (0,R)} \frac{K_\varepsilon(x,y)}{y} \le \overline{\omega}_R(M), \quad \text{for } M > 0 \quad \text{and} \quad \lim_{M \to +\infty} \overline{\omega}_R(M) = 0. \tag{42}$$

Thus,

$$\left| \frac{d}{dt} \int_0^\infty f_\varepsilon(t,x)\,\psi(x)\,dx \right| \le M^2 \left[3\|K\|_{L^\infty((0,R+1)^2)} + 4 \sup_{y \ge R} \overline{\omega}_{R+1}(y) \right] \|\psi\|_{W^{1,\infty}},$$

which, together with Lemma 7, yields (41). □

3.2 Convergence

Lemma 10 *There exist a non-negative function f and a subsequence of (f_ε) (not relabelled) such that, for every $T \in (0,+\infty)$,*

$$f \in L^\infty(0,T;L^1_1(\mathbb{R}_+)) \qquad \textit{and} \qquad f_\varepsilon \longrightarrow f \quad \textit{in} \quad \mathcal{C}([0,T]; w-L^1(\mathbb{R}_+)). \tag{43}$$

Proof. Let $T>0$. Due to [11, Theorem 1.3.2], it suffices to check that

$$\text{the family } (f_\varepsilon) : [0,T] \to L^1(\mathbb{R}_+) \text{ is weakly equicontinuous,} \tag{44}$$

$$\text{the set } \{f_\varepsilon(t), \varepsilon \in (0,1)\} \text{ is weakly relatively compact in } L^1(\mathbb{R}_+), \tag{45}$$

for every $t \in [0,T]$, to conclude that (f_ε) is relatively sequentially compact in $\mathcal{C}([0,T]; w-L^1(\mathbb{R}_+))$.

We first prove (45). Since $f^{in} \in L^1(\mathbb{R}_+)$, a refined version of the de la Vallée Poussin theorem [12, 13] ensures the existence of a function ϕ fulfilling the assumptions of Lemma 8 and such that

$$\lim_{r\to+\infty} \frac{\phi(r)}{r} = 0 \qquad \text{and} \qquad \int_0^\infty \phi(f^{in})(x)\,dx < +\infty.$$

We then infer from Lemmas 6, 7 and 8 that

$$\sup_{\varepsilon\in(0,1)} \sup_{t\in[0,T]} \left\{ \int_0^\infty f_\varepsilon(t,x)\,(1+x)\,dx + \int_0^\infty \phi(f_\varepsilon(t,x))\,dx \right\} < +\infty, \tag{46}$$

whence (45) by the Dunford-Pettis theorem.

We now turn our attention to (44). Let $\varphi \in L^\infty(\mathbb{R}_+)$. There exists a sequence of functions (φ_k) in $\mathcal{C}^1_c(\mathbb{R}_+)$ such that

$$\varphi_k \quad \longrightarrow \quad \varphi \quad a.e. \text{ in } \mathbb{R}_+, \tag{47}$$

$$\|\varphi_k\|_{L^\infty} \quad \leq \quad \|\varphi\|_{L^\infty} \tag{48}$$

We fix $\eta \in (0,1)$. From (46), we deduce the existence of some real $\delta(\eta)>0$ such that, for any measurable subset E of $\mathbb{R}_+$,

$$\sup_{\varepsilon\in(0,1)} \sup_{t\in[0,T]} \int_E f_\varepsilon(t,x)\,dx \leq \eta, \tag{49}$$

as soon as $\text{meas}(E) \leq \delta(\eta)$. Moreover, the Egorov theorem and (47) imply the existence of a measurable subset E_η of $[0,1/\eta]$ such that

$$\text{meas}\,(E_\eta) \leq \delta(\eta) \qquad \text{and} \qquad \lim_{k\to+\infty} \sup_{[0,1/\eta]\setminus E_\eta} |\varphi_k - \varphi| = 0.$$

Consequently, for all $t \in (0,T)$, $h \in (-t, T-t)$ and $R \in [0, 1/\eta]$, we have

$$\begin{aligned}\left| \int_0^\infty [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\, \varphi(x)\, dx \right| &\le \left| \int_0^R [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\, \varphi_k(x)\, dx \right| \\ &+ \left| \int_0^R [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\, [\varphi(x) - \varphi_k(x)]\, dx \right| \\ &+ \left| \int_R^\infty [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\, \varphi(x)\, dx \right|.\end{aligned}$$

Thus, by the definition of $\delta(\eta)$, E_η and φ_k, we deduce from Lemmas 6 and 7 that

$$\begin{aligned}\left| \int_0^\infty [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\, \varphi(x)\, dx \right| &\le \left| \int_t^{t+h} \frac{d}{ds} \left(\int_0^R f_\varepsilon(s,x)\, \varphi_k(x)\, dx \right) ds \right| \\ &+2\, M \sup_{[0,R]\setminus E_\eta} |\varphi_k - \varphi| + 4\, \|\varphi\|_{L^\infty}\, \eta \\ &+ \frac{4\, \|\varphi\|_{L^\infty}\, M}{R}.\end{aligned}$$

Then, Lemma 9 ensures that

$$\begin{aligned}\sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T)} \left| \int_0^\infty [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\, \varphi(x)\, dx \right| &\le |h|\, C(\varphi_k) + 2\, M \sup_{[0,R]\setminus E_\eta} |\varphi_k - \varphi| \\ &+4\, \|\varphi\|_{L^\infty}\, \eta + \frac{4\, \|\varphi\|_{L^\infty}\, M}{R}.\end{aligned}$$

We let $h \to 0$ and obtain, thanks to Lemma 9, that

$$\begin{aligned}&\limsup_{h\to 0} \sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T)} \left| \int_0^\infty [f_\varepsilon(t+h,x) - f_\varepsilon(t,x)]\, \varphi(x)\, dx \right| \\ &\qquad \le 2\, M \sup_{[0,R]\setminus E_\eta} |\varphi_k - \varphi| + 4\, \|\varphi\|_{L^\infty}\, \eta + \frac{4\, \|\varphi\|_{L^\infty}\, M}{R}.\end{aligned}$$

We now pass to the successive limits $k \to +\infty$, $\eta \to 0$ and $R \to +\infty$ and deduce that (44) holds. Therefore, the proof of Lemma 10 is complete. □

We now check that the function f constructed in Lemma 10 is a weak solution to the OHS equation. We consider $\varphi \in \mathcal{D}(\mathbb{R}_+)$ and define φ_ε by (9). It is easily checked that f_ε satisfies, for every $t \in (0, \infty)$,

$$\begin{aligned}&\int_0^\infty f_\varepsilon(t,x)\, \varphi_\varepsilon(x)\, dx - \int_0^\infty f_\varepsilon(0,x)\, \varphi_\varepsilon(x)\, dx \\ &\qquad = \int_0^t \int_0^\infty \int_0^{r_\varepsilon(x)} K_\varepsilon(x,y)\, f_\varepsilon(s,x) f_\varepsilon(s,y)\, [\, y\, D_\varepsilon(\varphi_\varepsilon(x)) - \varphi_\varepsilon(y)]\, dy\, dx\, ds. \quad (50)\end{aligned}$$

It remains now to pass to the limit as $\varepsilon \to 0$ in (50). For that purpose, we need some convergence results for φ_ε and K_ε.

Lemma 11 *The sequences φ_ε and K_ε defined by (9) and (27) satisfy, for every $R>0$, the following properties:*

$$\|\varphi_\varepsilon\|_{L^\infty} \le \|\varphi\|_{L^\infty} \quad \text{and} \quad \varphi_\varepsilon \longrightarrow \varphi \;\text{ strongly in } L^\infty(\mathbb{R}_+), \tag{51}$$
$$\|D_\varepsilon(\varphi_\varepsilon)\|_{L^\infty} \le \|\varphi\|_{W^{1,\infty}} \quad \text{and} \quad D_\varepsilon(\varphi_\varepsilon) \longrightarrow \partial_x\varphi \;\text{ strongly in } L^\infty(\mathbb{R}_+), \tag{52}$$
$$\|K_\varepsilon\|_{L^\infty((0,R)^2)} \le \|K\|_{L^\infty((0,R+1)^2)} \quad \text{and} \quad K_\varepsilon \longrightarrow K \quad \text{a.e. on } \mathbb{R}_+^2. \tag{53}$$

Proof. Let $x \in \mathrm{supp}(\varphi)$. For ε sufficiently small depending only on $\mathrm{supp}(\varphi)$, there is $i \ge 1$ such that $x \in \Lambda_i^\varepsilon$. Then,

$$\begin{aligned}
|D_\varepsilon(\varphi_\varepsilon)(x) - \partial_x\varphi(x)| &= \left| \frac{1}{\varepsilon}\int_{\Lambda_i^\varepsilon} \left[\frac{\varphi(z+\varepsilon)-\varphi(z)}{\varepsilon} - \partial_x\varphi(x) \right] dz \right| \\
&= \left| \frac{1}{\varepsilon^2}\int_{\Lambda_i^\varepsilon}\int_z^{z+\varepsilon} [\partial_x\varphi(w) - \partial_x\varphi(x)]\, dw\, dz \right| \\
&\le 2\varepsilon \|\varphi\|_{W^{2,\infty}},
\end{aligned}$$

whence (52). Similar calculations lead to (51). As for (53), it readily follows from the definition (27) of K_ε. □

We next recall the classical following lemma (see, e.g. [14, Lemma A.2] for a proof).

Lemma 12 *Let U be an open set of $\mathbb{R}^m$, $m \ge 1$, and consider two sequences (v_n) in $L^1(U)$ and (w_n) in $L^\infty(U)$. We suppose that there exist v in $L^1(U)$, w in $L^\infty(U)$ and $C>0$ such that*

$$\begin{aligned}
&v_n \rightharpoonup v \;\text{ in } L^1(U), \\
&\|w_n\|_{L^\infty} \le C \quad \text{and} \quad w_n \to w \quad \text{a.e. in } U.
\end{aligned}$$

Then

$$\lim_{n\to+\infty} \|v_n(w_n - w)\|_{L^1} = 0 \qquad \text{and} \qquad v_n\, w_n \rightharpoonup v\, w \quad \text{in} \quad L^1(U).$$

We are now in a position to pass to the limit in (50). Let $\varphi \in \mathcal{D}(\mathbb{R}_+)$ with $\mathrm{supp}(\varphi) \subset [0, L-2]$, for some $L>2$, and define φ_ε by (9). Let $T>0$ and $R>L$. On the one hand, it follows from Lemma 10 by classical arguments that

$$f_\varepsilon(t,x)\, f_\varepsilon(t,y) \longrightarrow f(t,x)\, f(t,y) \quad \text{ in } C([0,T]; w - L^1((0,R)^2)).$$

On the other hand, the definition (11) of r_ε ensures that $\mathbf{1}_{[0,r_\varepsilon(x)]} \longrightarrow \mathbf{1}_{[0,x]}$ for a.e. $x \in \mathbb{R}_+$, which together with Lemma 11, implies that

$$\begin{aligned}
&K_\varepsilon(x,y)\, [\, y\, D_\varepsilon(\varphi_\varepsilon)(x) - \varphi_\varepsilon(y)]\, \mathbf{1}_{[0,r_\varepsilon(x)]}(y) \\
&\qquad \longrightarrow \quad K(x,y)\, [y\, \partial_x\varphi(x) - \varphi(y)]\, \mathbf{1}_{[0,x]}(y) \qquad a.e. \text{ in } (0,R)^2.
\end{aligned}$$

Owing to the bounds on φ_ε, $D_\varepsilon(\varphi_\varepsilon)$ and K_ε in Lemma 11, we may apply Lemma 12 to obtain that

$$\int_0^T\int_0^R\int_0^R K_\varepsilon(x,y)\, f_\varepsilon(t,x)\, f_\varepsilon(t,y)\, [\, y\, D_\varepsilon(\varphi_\varepsilon)(x) - \varphi_\varepsilon(y)]\, \mathbf{1}_{[0,r_\varepsilon(x)]}(y)\, dy\, dx\, dt$$
$$\overset{\varepsilon\to 0}{\longrightarrow} \int_0^T\int_0^R\int_0^R K(x,y)\, f(t,x)\, f(t,y)\, [y\, \partial_x\varphi(x) - \varphi(y)]\, \mathbf{1}_{[0,x]}(y)\, dy\, dx\, dt.$$

Also, since $\mathrm{supp}(\varphi) \subset [0, R-2]$,

$$\begin{aligned}\iint_{\mathbb{R}_+^2\setminus[0,R]^2} K_\varepsilon(x,y)\, f_\varepsilon(t,x)\, f_\varepsilon(t,y)\, y\, D_\varepsilon(\varphi_\varepsilon)(x)\, \mathbf{1}_{[0,r_\varepsilon(x)]}(y)\, dy\, dx &= 0,\\ \iint_{\mathbb{R}_+^2\setminus[0,R]^2} K(x,y)\, f(t,x)\, f(t,y)\, y\, \partial_x\varphi(x)\, \mathbf{1}_{[0,x]}(y)\, dy\, dx &= 0.\end{aligned}$$

Finally, it follows from (15) and (42) that

$$\begin{aligned}&\left|\iint_{\mathbb{R}_+^2\setminus[0,R]^2} K_\varepsilon(x,y)\, f_\varepsilon(t,x)\, f_\varepsilon(t,y)\, \varphi_\varepsilon(y)\, \mathbf{1}_{[0,r_\varepsilon(x)]}(y)\, dy\, dx\right|\\ &\qquad\le \left|\int_R^\infty dx \int_0^L K_\varepsilon(x,y)\, f_\varepsilon(t,x)\, f_\varepsilon(t,y)\, \varphi_\varepsilon(y)\, dy\right|\\ &\qquad\qquad\le C\, M^2\, \|\varphi\|_{L^\infty} \sup_{x\ge R} \overline{\omega}_L(x),\end{aligned}$$

and

$$\begin{aligned}&\left|\iint_{\mathbb{R}_+^2\setminus[0,R]^2} K(x,y)\, f(t,x)\, f(t,y)\, \varphi(y)\, \mathbf{1}_{[0,x]}(y)\, dy\, dx\right|\\ &\qquad\le \left|\int_R^\infty dx \int_0^L K(x,y)\, f(t,x)\, f(t,y)\, \varphi(y)\, dy\right|\\ &\qquad\qquad\le C\ M^2\, \|\varphi\|_{L^\infty} \sup_{x\ge R} \omega_L(x).\end{aligned}$$

Therefore, they both tend to 0 as $R \to +\infty$, uniformly with respect to ε.

It remains now to let $\varepsilon \to 0$ in the first two terms of (50). It readily follows from Lemmas 7, 10 and 11 that

$$\int_0^\infty f_\varepsilon(t,x)\, \varphi_\varepsilon(x)\, dx \longrightarrow \int_0^\infty f(t,x)\, \varphi(x)\, dx,$$

for every $t > 0$. Moreover,

$$f_\varepsilon(0) \longrightarrow f^{in} \qquad \text{in} \quad L^1(\mathbb{R}_+), \tag{54}$$

whence, by Lemma 11,

$$\int_0^\infty f_\varepsilon(0,x)\, \varphi_\varepsilon(x)\, dx \longrightarrow \int_0^\infty f^{in}(x)\, \varphi(x)\, dx. \tag{55}$$

We thereby obtain that f satisfies (17) and is, consequently, a weak solution to the OHS equation.

To justify (54), we first observe that, for $f^{in} \in W^{1,1}(\mathbb{R}_+)$, we have

$$\|f_\varepsilon(0,.) - f^{in}\|_{L^1} \le \varepsilon \, \|f^{in}\|_{W^{1,1}},$$

whence (54), for $f^{in} \in W^{1,1}(\mathbb{R}_+)$. The general case for $f^{in} \in L^1(\mathbb{R}_+)$ then follows by a density argument, since

$$\|f_\varepsilon(0,.) - g_\varepsilon(0,.)\|_{L^1} \le \|f^{in} - g^{in}\|_{L^1},$$

for every $f^{in}, g^{in} \in L^1(\mathbb{R}_+)$.

4 The dOHS equation

4.1 Proof of Proposition 4

We are here concerned with the Cauchy problem (4)-(6) where the discrete coefficients $K_{i,j}$ satisfy (28) and the initial data $c^{in} = (c_i^{in})_{i\ge 1}$ satisfies (29). We proceed as in [8, 9]: we first approximate the dOHS equation by a system of ordinary differential equations.

Let $N \ge 3$ be a positive integer. We consider the following system of N ordinary differential equations:

$$\frac{dc_i^N}{dt} = Q_i^N(c^N), \qquad \text{in } (0,+\infty), \tag{56}$$

$$c_i^N(0) = c_i^{in}, \tag{57}$$

for $i \in \{1,\ldots,N\}$, where $c^N = (c_i^N)_{1\le i\le N}$ and

$$Q_i^N(c^N) = c_{i-1}^N \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j^N - c_i^N \sum_{j=1}^{i} j\, K_{i,j}\, c_j^N - \sum_{j=i}^{N} K_{i,j}\, c_i^N\, c_j^N. \tag{58}$$

We first prove the well-posedness of (56), (57).

Lemma 13 *For each $N \ge 3$, there exists a unique non-negative solution $c^N = (c_i^N)_{1\le i\le N}$ in $\mathcal{C}^1([0,+\infty),\mathbb{R}^N)$ to the system (56)-(58). Moreover, we have*

$$\sum_{i=1}^N i\, c_i^N(t) \le \sum_{i=1}^N i\, c_i^{in}, \qquad t \in [0,+\infty). \tag{59}$$

Proof. Consider $c^{in,N} = (c_i^{in,N}) \in \mathbb{R}^N$. Since Q^N is a locally Lipschitz continuous function, the Cauchy-Lipschitz theorem ensures the existence of a unique maximal solution $c^N = (c_i^N)_{1\le i\le N} \in \mathcal{C}^1([0,t^+(c^{in,N}));\mathbb{R}^N)$ to (56)-(58), where either $t^+(c^{in,N}) = +\infty$, or $t^+(c^{in,N}) < +\infty$ and

$$\lim_{t\to t^+(c^{in,N})} \sum_{i=1}^N |c_i^N(t)| = +\infty.$$

Now, let $c^{in,N} \in [0,+\infty)^N$. Then $c^{in,N} + t\, Q^N(c^{in,N}) \in [0,+\infty)^N$ if t satisfies

$$t \left((N+1) \times \left(\sup_{1\le i,j\le N} K_{i,j} \right) \times \sum_{j=1}^{N} c_j^{in,N} \right) \le 1.$$

Consequently, $\mathrm{dist}(c^{in,N} + t\, Q^N(c^{in,N}), [0,+\infty)^N) = 0$ for t small enough and thus,

$$\liminf_{t\to 0^+} \quad t^{-1}\ \mathrm{dist}\left(c^{in,N} + t\, Q^N(c^{in,N}), [0,+\infty)^N \right) = 0,$$

which corresponds to the *subtangent condition*. Therefore, [15, Theorem 16.5] ensures that, for each $c^{in,N} \in [0,+\infty)^N$, the corresponding maximal solution $c^N = (c_i^N)_{1\le i\le N}$ to (56)-(58) is non-negative on $[0, t^+(c^{in,N}))$.

Besides, we note that, for every $c \in \mathbb{R}^N$,

$$\sum_{i=1}^{N} i\, Q_i^N(c) = -(N+1) c_N \sum_{j=1}^{N} K_{N,j}\, j c_j.$$

Consequently, we have, for each $t \in [0, t^+(c^{in,N}))$,

$$0 \le \sum_{i=1}^{N} |c_i^N(t)| = \sum_{i=1}^{N} c_i^N(t) \le \sum_{i=1}^{N} i\, c_i^N(t) \le \sum_{i=1}^{N} i\, c_i^{in,N} < +\infty,$$

where $c^{in,N} \in [0,+\infty)^N$ and c^N denotes the corresponding maximal solution to (56)-(58). This implies that $t^+(c^{in,N}) = +\infty$ for each $c^{in,N} \in [0,+\infty)^N$ and completes the proof of Lemma 13. □

It remains now to pass to the limit in (56)-(58). To this end, we need some compactness property. By (59), we already know that $(c_i^N)_{N\ge i}$ is bounded for each $i \ge 1$. We next prove the time equicontinuity of $(c_i^N)_{N\ge i}$.

Lemma 14 *Let $i \ge 1$. There exists a constant γ_i, depending only on $\sum_{k=1}^{\infty} k\, c_k^{in}$ and i such that, for each $N \ge i$,*

$$\left| \frac{dc_i^N}{dt} \right| \le \gamma_i, \qquad t \in [0,+\infty). \tag{60}$$

Proof. Due to (28), we set, for each $i \ge 1$,

$$\kappa_i := \sup_j \frac{K_{i,j}}{j} < +\infty.$$

Then equations (56), (58) and (59) imply that

$$\begin{aligned} \left| \frac{dc_i^N}{dt} \right| &\le \kappa_{i-1}\, (i-1)\, c_{i-1}^N \sum_{j=1}^{N} j\, c_j^N + \kappa_i\, i\, c_i^N \sum_{j=1}^{N} j\, c_j^N + \kappa_i\, c_i^N \sum_{j=1}^{N} j\, c_j^N \\ &\le (\kappa_{i-1} + 2\kappa_i) \left(\sum_{j=1}^{\infty} j c_j^{in} \right)^2 . \end{aligned}$$

□

Gathering Lemmas 13 and 14, we deduce from the Arzela-Ascoli theorem that there exist a function $c = (c_i)_{i\geq 1}$ and a subsequence of $(c_i^N)_{N\geq i}$, not relabelled, such that

$$c_i^N \longrightarrow c_i \quad \text{in} \quad \mathcal{C}([0,T]), \tag{61}$$

for all $i \geq 1$ and $T > 0$. Then, for each $i \geq 1$, c_i is a non-negative function on $[0,\infty)$ and

$$\sum_{i=1}^{\infty} i\, c_i(t) \leq \sum_{i=1}^{\infty} i\, c_i^{in},$$

for every $t \geq 0$. Consequently, we have

$$\sum_{j=i}^{\infty} K_{i,j}\, c_j(t) \leq \left(\sup_{j\geq i} \frac{K_{i,j}}{j}\right) \times \sum_{j=1}^{\infty} j\, c_j^{in} \leq \kappa_i \sum_{j=1}^{\infty} j\, c_j^{in}, \qquad t \in (0,\infty),$$

whence

$$\sum_{j=i}^{\infty} K_{i,j}\, c_j \in L^1(0,t) \quad \text{for every} \quad t \in (0,\infty).$$

Let $i \geq 1$. We now infer from (61) that, for every $t \geq 0$,

$$\int_0^t c_{i-1}^N(s) \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j^N(s)\, ds \overset{N\to+\infty}{\longrightarrow} \int_0^t c_{i-1}(s) \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j(s)\, ds,$$

$$\int_0^t c_i^N(s) \sum_{j=1}^{i} j\, K_{i,j}\, c_j^N(s)\, ds \overset{N\to+\infty}{\longrightarrow} \int_0^t c_i(s) \sum_{j=1}^{i} j\, K_{i,j}\, c_j(s)\, ds.$$

It remains only to pass to the limit in the last sum of (58). To this end, we fix $M \geq i$. For $N > M$, we have

$$\left| \int_0^t \left(\sum_{j=i}^{N} K_{i,j}\, c_i^N(s)\, c_j^N(s) - \sum_{j=i}^{\infty} K_{i,j}\, c_i(s)\, c_j(s) \right) ds \right|$$

$$\leq \left| \int_0^t \sum_{j=i}^{M-1} K_{i,j} \left[c_i^N(s)\, c_j^N(s) - c_i(s)\, c_j(s)\right] ds \right| \tag{62}$$

$$+ \left| \int_0^t \sum_{j=M}^{N} K_{i,j}\, c_i^N(s)\, c_j^N(s)\, ds \right| + \left| \int_0^t \sum_{j=M}^{\infty} K_{i,j}\, c_i(s)\, c_j(s)\, ds \right|, \tag{63}$$

for every $t \geq 0$. By (61), expression (62) tends to 0 as $N \to +\infty$. As for (63), we need the growth assumption of (28):

$$\int_0^t \sum_{j=M}^{N} K_{i,j}\, c_i^N(s)\, c_j^N(s)\, ds \leq \sup_{j\geq M} \frac{K_{i,j}}{j} \int_0^t \sum_{j=M}^{N} c_i^N(s)\, j\, c_j^N(s)\, ds \leq t \left(\sum_{j=1}^{\infty} j\, c_j^{in} \right)^2 \sup_{j\geq M} \frac{K_{i,j}}{j},$$

and, similarly,

$$\int_0^t \sum_{j=M}^{\infty} K_{i,j}\, c_i(s)\, c_j(s)\, ds \le t \left(\sum_{j=1}^{\infty} j\, c_j^{in} \right)^2 \sup_{j\ge M} \frac{K_{i,j}}{j},$$

for every $t \ge 0$. Letting first $N \to +\infty$ and then $M \to +\infty$, we thus obtain that c satisfies (4)-(6). The function c is thus a solution to the dOHS equation in the sense of Definition 3.

4.2 Proof of Proposition 5

We finally show that the dOHS equation propagates perturbations with an infinite speed.

By (4), the solution c satisfies, for all $i \ge 1$ and $t \in (0,T)$,

$$c_i(t) = c_i^{in} \exp\left(- \int_0^t E_i(s)\, ds \right) + \int_0^t \exp\left(- \int_s^t E_i(\sigma)\, d\sigma \right) c_{i-1}(s)\, F_i(s)\, ds, \qquad (64)$$

where

$$E_i(s) = \sum_{j=1}^{i} j\, K_{i,j}\, c_j(s) + \sum_{j=i}^{\infty} K_{i,j}\, c_j(s) \qquad \text{and} \qquad F_i(s) = \sum_{j=1}^{i-1} j\, K_{i-1,j}\, c_j(s),$$

for every $s \in (0,T)$.

Let us assume that, contrary to our claim, $c_r(\tau) = 0$, for some $r \ge k$ and some $\tau \in (0,T)$. By (64), $c_r(\tau)$ is the sum of two non-negative terms and thus

$$c_r^{in} = 0 \qquad \text{and} \qquad c_{r-1}\, F_r \equiv 0 \quad \text{on } [0,\tau].$$

If $r = 1$, then $k = 1$ and we have $c_1^{in} = 0$, which contradicts the assumption of Proposition 5. If $r > 1$, we have in particular that $(r-1)K_{r-1,r-1}c_{r-1}^2 \equiv 0$ on $[0,\tau]$, whence $c_{r-1} \equiv 0$ on $[0,\tau]$. Consequently, the assumption $c_r(\tau) = 0$ implies that

$$c_r^{in} = 0 \qquad \text{and} \qquad c_{r-1}(\tau) = 0.$$

By induction, we deduce that $c_i^{in} = 0$ for every $i \le r$. In particular, this leads to a contradiction for $i = k$.

5 Numerical simulations

In this section, we perform numerical experiments in order to illustrate the convergence in Theorem 2. We consider the particular case where $K \equiv 1$ on $\mathbb{R}_+^2$ and the initial data is given by

$$F_M^{in} = \frac{2}{M}\, \mathbf{1}_{[0,M]} \quad \text{on } \mathbb{R}_+, \qquad (65)$$

for some $M > 0$. In that case, there is an explicit solution to the OHS equation, which reads

$$F_M(t,x) = \frac{2}{M(1+t)^2}\, \mathbf{1}_{[0,M]} \left(\frac{x}{1+t}\right), \qquad (t,x) \in \mathbb{R}_+^2.$$

The computational domain is chosen to be $[0,10]$ and we set $M = 3$. For any $\varepsilon \in (0,1)$, we define the initial data $c^{in,\varepsilon} = (c_i^{in,\varepsilon})_{i\geq 1}$ for the dOHS equation by (22), where Λ_i^ε is given by (8). We next consider the system of ordinary differential equations

$$\frac{dc_i^m}{dt} = \varepsilon \left(c_{i-1}^m \sum_{j=1}^{i-1} j\, c_j^m - c_i^m \sum_{j=1}^{i} j\, c_j^m - \sum_{j=i}^{m} c_i^m\, c_j^m \right) \qquad \text{in } (0,+\infty), \tag{66}$$

$$c_i^m(0) = c_i^{in}, \tag{67}$$

for $i \in \{1,\ldots,m\}$, where $m = m(\varepsilon) = [10/\varepsilon - 1/2]$ corresponds to the number of cells Λ_i^ε included in the interval $[0,10]$. We next use a Matlab ODE solver to obtain a solution to (66), (67) on some time interval $[0, t_{max}]$. The approximated solution f_ε is then given by

$$f_\varepsilon(t,x) = \sum_{i=1}^{m} c_i^m(t)\, \chi_i^\varepsilon(x), \qquad (t,x) \in [0,t_{max}] \times [0,10].$$

The plot of the exact solution F_3 and the approximated solutions f_ε, for $\varepsilon = 0.05$, $\varepsilon = 0.01$ and $\varepsilon = 0.005$ is reported in Figure 1 at two different times (for $t_{max} = 3$), while the time evolution of the L^1 relative error

$$t \longmapsto \frac{\|F_3 - f_\varepsilon\|_{L^1}}{\|F_3\|_{L^1}}(t),$$

is plotted in Figure 2. Both figures illustrate the L^1 convergence of f_ε to F_3 as $\varepsilon \to 0$. We point out that the error is concentrated in the neighbourhood of the discontinuity of F_3 (see Figure 1 (a)), which was expected since the upwind difference scheme is diffusive and diffusion smears out the discontinuities. A further comment in that direction is that, in Figure 2, the L^1 relative error decreases for $t \in [7/3, 5/2]$, which can be explained by the fact that the discontinuity of F_3 leaves the computational domain at $t = 7/3$. Afterwards, we have $F_3(t) \equiv 2/(3(1+t)^2)$ on $[0,10]$ and the remaining error is mainly due to the truncation of the computational domain. Another source of error comes from Proposition 5, which states that, contrary to F_3, f_ε is not compactly supported. Consequently, our approximation induces some errors outside the support of the exact solution. Finally, from $t = 7/3$, the support of the exact solution F_3 is no more included in the computational domain $[0,10]$ and the approximation will be less and less reliable as t increases.

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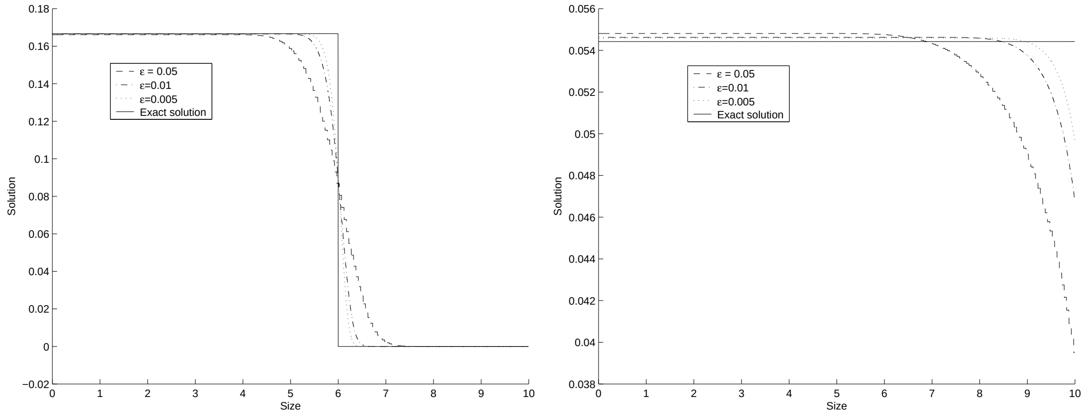

(a) (b)

Figure 1: Convergence of solutions to the dOHS equations towards the solution to the OHS equation with initial data (65) at times $t = 1$ (a) and $t = 2.5$ (b) for $M = 3$

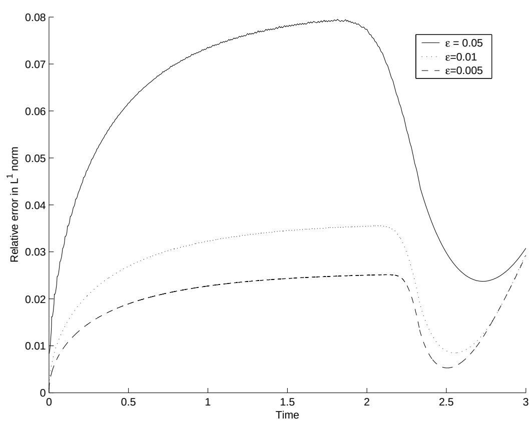

Figure 2: Evolution of the relative error in L^1 norm for $M = 3$

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