Well-posedness for the spatially homogeneous Landau-Fermi-Dirac equation for hard potentials

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Abstract

We study the Cauchy problem for the spatially homogeneous Landau equation for Fermi-Dirac particles, in the case of hard and Maxwellian potentials. We establish existence and uniqueness of a weak solution for a large class of initial data.

1 Introduction

Kinetic theory aims at modelling a gas or a plasma when one is interested rather in the statistical properties of the gas than in the state of each gas particle. The evolution of the gas is then described by a distribution function $f = f(t, x, v) \geq 0$ which represents the (local) density of particles with velocity $v \in \mathbb{R}^3$ at position $x \in \mathbb{R}^3$ and time $t \in \mathbb{R}_+ := [0, +\infty[$.

In the absence of interactions (or collisions) between particles, the evolution of f is given by the free transport equation. When the effect of collisions is included, f satisfies the celebrated Boltzmann equation or related models [3, 4, 5, 20]. In particular, while the Boltzmann equation is valid for neutral particles or weakly ionised plasmas, the modelling of completely ionised plasmas introduces a new model, the Landau equation, which is obtained as a limit of the Boltzmann equation when grazing collisions prevail (cf. [5, 8, 9, 20]). Also quantum effects such as the Pauli exclusion principle should sometimes be taken into account and both the Boltzmann and Landau equations have to be modified accordingly in that case [5, 7, 20]. We also mention that a Landau equation with Fermi statistics arises in the modelling of self gravitating particles [6, 16].

In this paper, we study a modified Landau equation accounting for the Pauli exclusion principle which reads:

$$\partial_t f + v \cdot \nabla_x f = Q_L(f),$$

where

$$Q_L(f) = \nabla_v \cdot \int \Psi(v - v_*)\, \Pi(v - v_*) \Big\{ f_*(1 - \delta f_*) \nabla f - f(1 - \delta f) \nabla f_* \Big\}\, dv_*,$$

with $\delta = 1$, $f = f(t, v)$, $f_* = f(t, v_*)$, $\Pi(z)$ denotes the orthogonal projection on $(\mathbb{R}z)^\perp$,

$$\Pi_{i,j}(z) = \delta_{i,j} - \frac{z_i z_j}{|z|^2}, \qquad 1 \leq i, j \leq 3,$$

and Ψ is a function such as $\Psi(z) = |z|^{2+\gamma}$, $-3 \leq \gamma \leq 1$. The choice $\Psi(z) = |z|^{2+\gamma}$ corresponds to inverse power law potentials, among which we distinguish the Coulomb potential ($\gamma = -3$), soft potentials ($-3 < \gamma < 0$), the Maxwellian potential ($\gamma = 0$) and hard potentials ($0 < \gamma \leq 1$). We recall here that the Coulomb potential is however the only one to have a physical relevance.

Taking $\delta = 0$ in $Q_L(f)$ corresponds to the classical Landau equation, while the Landau-Fermi-Dirac (LFD) equation and the Landau-Bose-Einstein (LBE) equation correspond to $\delta = 1$ and $\delta = -1$, respectively. Only the case $\delta = 1$ will be considered herein and our aim is to investigate the existence and uniqueness of weak solutions to the LFD equation in a spatially homogeneous setting, that is $f = f(t, v)$ and satisfies

$$\partial_t f = Q_L(f), \tag{1.1}$$

with $\delta = 1$. We point out that the Pauli exclusion principle implies that a solution to (1.1) must satisfy $0 \leq f \leq 1$.

While the classical Boltzmann and Landau equations have been the subject of several papers (see [3, 4, 11, 30] for the Boltzmann equation and [2, 10, 15, 29] for the Landau equation, and the references therein), fewer studies have been devoted to the Boltzmann-Fermi-Dirac (BFD) equation and to the LFD equation. Concerning the former, the spatially inhomogeneous Cauchy problem has been studied in [1, 12, 22] for cross sections satisfying Grad's cut-off assumption. In a spatially homogeneous setting, existence of solutions to the BFD equation is investigated in [13, 24] for more realistic cross sections, and their large time behaviour as well [13, 24, 25]. To our knowledge, the problem of existence and uniqueness of solutions to the LFD equation has not been yet considered, and the only works on this model concern a formal derivation from the BFD equation in the grazing collisions limit [7] and a spectral analysis of its linearization near an equilibrium [18]. Therefore, our purpose is to investigate the well-posedness of the Cauchy problem for the LFD equation in a spatially homogeneous setting for hard or Maxwellian potentials. As already mentioned, the Pauli exclusion principle implies that solutions to the LFD equation should satisfy the L^∞-bound $0 \leq f \leq 1$. On the one hand, this L^∞-bound simplifies the analysis in comparison to the classical Landau equation where only a bound in $L \log L$ is available. On the other hand, the term $f(1 - \delta f)$ is nonlinear for $\delta = 1$ and requires strong compactness arguments to be handled (weak compactness is sufficient for the classical Landau equation where $\delta = 0$, since the term $f(1 - \delta f) = f$ is linear in that case).

We now describe the contents of the paper. We set notations and state our main results in the next section: existence, propagation of moments, uniqueness (Theorem 2.2), ellipticity of $Q_L(f)$ (Proposition 2.3). *A priori* estimates are gathered in Section 3 and are used in Section 4 to prove the existence of a solution to the LFD equation. Finally, the uniqueness result stated in Theorem 2.2 is proved in Section 5.

2 Main results

We first introduce some notations and definitions. For $s \in \mathbb{R}$, $p \geq 1$ and $k \in \mathbb{N}$, we set

$$\begin{aligned} L^p_{2s}(\mathbb{R}^3) &:= L^p\Big(\mathbb{R}^3; \big(1+|v|^2\big)^s\, dv\Big), \\ \|f\|^p_{L^p_{2s}} &= \int \big|f(v)\big|^p \big(1+|v|^2\big)^s\, dv, \\ \|f\|^2_{H^k_{2s}} &= \sum_{0\leq|\alpha|\leq k} \int \big|\partial^\alpha_x f(v)\big|^2 \big(1+|v|^2\big)^s\, dv, \end{aligned}$$

where $\alpha = (i_1, i_2, i_3) \in \mathbb{N}^3$, $|\alpha| = i_1 + i_2 + i_3$ and $\partial^\alpha_x f = \partial^{i_1}_1 \partial^{i_2}_2 \partial^{i_3}_3 f$.

For $s \geq 0$ and $f \in L^1_{2s}(\mathbb{R}^3)$, we denote by $M_{2s}(f)$ the moment of order $2s$ of f, that is

$$M_{2s}(f) = \int \big|f(v)\big|\, |v|^{2s}\, dv.$$

For $(i,j) \in [\![1,3]\!]^2$, we define

$$\begin{aligned} a(z) = \big(a_{i,j}(z)\big)_{i,j} \quad &\text{with} \quad a_{i,j}(z) = |z|^{\gamma+2}\left(\delta_{i,j} - \frac{z_i z_j}{|z|^2}\right), \\ b_i(z) &= \sum_k \partial_k a_{i,k}(z) = -2\, z_i\, |z|^\gamma, \\ c(z) &= \sum_{k,l} \partial^2_{kl} a_{k,l}(z) = -2\,(\gamma+3)\, |z|^\gamma, \end{aligned}$$

and, when no confusion can occur, we write $\overline{A} = (\overline{A}_{i,j})$, $\overline{b} = (\overline{b}_i)$, $\overline{B} = (\overline{B}_i)$ with

$$\begin{aligned} \overline{b}_i &= b_i * f, & \overline{c} &= c * f, \\ \overline{A}_{i,j} &= a_{i,j} * \big(f(1-f)\big), & \overline{B}_i &= b_i * \big(f(1-f)\big). \end{aligned}$$

Otherwise, we use the notations $\overline{A}^f_{i,j}$, $\overline{b}^f_i$, $\overline{B}^f_i$, $\overline{c}^f$ instead of $\overline{A}_{i,j}$, $\overline{b}_i$, $\overline{B}_i$ and $\overline{c}$.

With these notations, the LFD equation can then be written alternatively under the form

$$\partial_t f = \nabla \cdot \big(\overline{A}\, \nabla f - \overline{b}\, f(1-f)\big), \tag{2.1}$$

or

$$\partial_t f = \sum_{i,j} \overline{A}_{i,j}\, \partial^2_{i,j} f + \big(\overline{B} - \overline{b}\,(1-2f)\big) \cdot \nabla f - \overline{c}\, f(1-f),$$

and is supplemented with the initial datum

$$f(0) = f_{in}, \tag{2.2}$$

where

$$f_{in} \in L^1_2(\mathbb{R}^3), \quad 0 \leq f_{in} \leq 1 \ \text{ a.e.} \quad \text{and} \quad f_{in} \not\equiv 0. \tag{2.3}$$

We note that the last assumption is not restrictive since when $f_{in} \equiv 0$, $f \equiv 0$ is a solution to (2.1), (2.2).

The usual *a priori* estimates are here available. Indeed, one can formally verify that solutions preserve mass and energy, namely

$$\forall t \geq 0, \qquad M_0(f)(t) = \int f(t,v)\,dv = \int f_{in}\,dv := M_{in}, \tag{2.4}$$

$$M_2(f)(t) = \int f(t,v)\,|v|^2\,dv = \int f_{in}\,|v|^2\,dv := E_{in}. \tag{2.5}$$

Moreover, introducing the entropy $S(f)$ for Fermi-Dirac particles defined by

$$S(f) = -\int \Big[f \ln f + (1-f)\ln(1-f)\Big]\,dv \geq 0,$$

one can see, still formally, that $t \longmapsto S(f)(t)$ is a non-decreasing function.

Definition 2.1 *Consider f_{in} satisfying (2.3). A weak solution to the LFD equation (2.1), (2.2) is a function f satisfying*

(i) $\quad f \in L^\infty(\mathbb{R}_+; L^1_2(\mathbb{R}^3)) \cap \mathcal{C}(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^3)), f(1-f) \in L^1_{loc}(\mathbb{R}_+; L^1_{2+\gamma}(\mathbb{R}^3))$;

(ii) $\quad 0 \leq f \leq 1$ *and* $f(0) = f_{in}$;

(iii) $\quad \forall t \geq 0, \quad \int f(t,v)\,|v|^2\,dv \leq \int f_{in}(v)\,|v|^2\,dv$;

(iv) $\quad \forall \varphi \in \mathcal{D}\big(\mathbb{R}^3\big), \forall s,t \geq 0$;

$$\int f(t,v)\varphi(v)\,dv - \int f(s,v)\varphi(v)\,dv$$
$$= \int_s^t d\tau \left[\sum_{i,j} \int \overline{A}_{i,j}\, f\, \partial^2_{i,j}\varphi\,dv + \int f\,\overline{B}\cdot\nabla\varphi\,dv + \int f(1-f)\,\overline{b}\cdot\nabla\varphi\,dv\right].$$

Our main result is the following.

Theorem 2.2 *Consider f_{in} satisfying (2.3) and assume further that $f_{in} \in L^1_{2s_0}(\mathbb{R}^3)$ for some $s_0 > 1$. Then, there exists a weak solution f to (2.1), (2.2) satisfying (2.4), (2.5) and*

$$f(1-f) \in L^1_{loc}\big(\mathbb{R}_+; L^1_{2s_0+\gamma}(\mathbb{R}^3)\big); \qquad f \in L^\infty_{loc}\big(\mathbb{R}_+; L^1_{2s_0}(\mathbb{R}^3)\big) \cap L^2_{loc}\big(\mathbb{R}_+; H^1_{2s_0}(\mathbb{R}^3)\big).$$

If we also suppose that $s_0 \geq 1 + \gamma/2$, $t \longmapsto S(f)(t)$ is a non-decreasing function and

$$\forall\, t \in \mathbb{R}_+, \qquad S_{in} := S(f_{in}) \leq S(f)(t) \leq E_{in} + \pi^{3/2}.$$

Moreover, for $2s_0 > 4\gamma + 11$, such a solution is unique.

The existence proof is adapted from that of [2, 10] and is performed in three steps: analysis of a regularized equation, uniform estimates and passage to the limit by a compactness argument. At this stage, we recall that, owing to the cubic nature of $Q_L(f)$, a weak compactness argument is not sufficient. Strong compactness is actually a consequence of the uniform ellipticity of the matrix $\overline{A}$ which we state now.

We fix $E_0 > 0$ and $S_0 > 0$ and denote by $\mathcal{Y}(E_0, S_0)$ the set of functions $f \in L^1_2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ such that $0 \leq f \leq 1$ a.e. and

$$M_2(f) \leq E_0, \qquad S(f) \geq S_0.$$

Proposition 2.3 *Let $f \in \mathcal{Y}(E_0, S_0)$. Then there exists a constant $K > 0$, depending only on γ, E_0 and S_0, such that*

$$\forall v \in \mathbb{R}^3,\ \forall \xi \in \mathbb{R}^3, \qquad \sum_{i,j} \overline{A}_{i,j}(v)\, \xi_i\, \xi_j \geq K \big(1 + |v|^2\big)^{\gamma/2}\, |\xi|^2.$$

As for the uniqueness proof, it follows the lines from that of [10], but the non-quadratic nature of $Q_L(f)$ requires the use of an embedding lemma for weighted Sobolev spaces.

3 *A priori* estimates

3.1 Uniform ellipticity

We first prove Proposition 2.3, and proceed as in [10, Proposition 4] for the Landau equation with some modifications. Indeed, for the classical Landau equation, the first step is a positive bound from below of $\|f\|_{L^1(B_R)}$ which is straightforward by (2.4) and (2.5) (B_R denotes the ball with center 0 and radius R). For the LFD equation, we need a positive bound from below of $\|f(1-f)\|_{L^1(B_R)}$ and we realize that the arguments of [10, Proposition 4] provide no information for velocities where f is close to 1. However, for such velocities, the needed information are to be found in the entropy.

Lemma 3.1 *There exist constants $\eta_* > 0$ and $R_* \geq 1$, depending only on E_0 and S_0, such that*

$$\forall f \in \mathcal{Y}(E_0, S_0), \qquad \int_{B_{R_*}} f(1-f)\, dv \geq \eta_* > 0.$$

Proof. Let $f \in \mathcal{Y}(E_0, S_0)$. For every $R \geq 1$, we have

$$S_0 \leq \int_{B_R} \big(f\,|\ln f| + (1-f)\,|\ln(1-f)|\big)\, dv + \int_{|v|\geq R} \big(f\,|\ln f| + (1-f)\,|\ln(1-f)|\big)\, dv. \quad (3.1)$$

Step 1. We first consider the integral over B_R. Let $\varepsilon, \alpha \in (0,1)$. Since

$$|\ln r| \leq (1-r)/\varepsilon \quad \text{if} \quad r \in (\varepsilon, 1), \quad (3.2)$$

$$|\ln(1-r)| \leq r/\varepsilon \quad \text{if} \quad r \in (0, 1-\varepsilon), \quad (3.3)$$

we deduce that

$$\begin{aligned} \int_{B_R} f\,|\ln f|\, dv &\leq \int_{B_R \cap \{f \geq \varepsilon\}} f\,|\ln f|\, dv + \int_{B_R \cap \{f \leq \varepsilon\}} f\,|\ln f|\, dv \\ &\leq \frac{1}{\varepsilon} \int_{B_R} f(1-f)\, dv + \varepsilon^\alpha \int_{B_R} f^{1-\alpha}\,|\ln f|\, dv, \end{aligned}$$

and, similarly,

$$\int_{B_R} (1-f)\,|\ln(1-f)|\, dv \leq \frac{1}{\varepsilon} \int_{B_R} f(1-f)\, dv + \varepsilon^\alpha \int_{B_R} (1-f)^{1-\alpha}\,|\ln(1-f)|\, dv.$$

As $r \longmapsto r^{1-\alpha}\,|\ln r|$ is bounded on $[0,1]$, we obtain, choosing $\varepsilon = R^{-4/\alpha}$,

$$\int_{B_R} \big(f\,|\ln f| + (1-f)\,|\ln(1-f)|\big)\, dv \;\leq\; 2R^{4/\alpha} \int_{B_R} f(1-f)\, dv + \frac{C_1(\alpha)}{R}\,. \quad (3.4)$$

Step 2. It remains now to consider the second integral of (3.1). On the one hand, thanks to the Hölder inequality and the boundedness of $r \longmapsto r^{\alpha}\,|\ln r|$ on $[0,1]$, we obtain

$$\begin{aligned}\int_{|v|\geq R} f\,|\ln f|\,dv &= \int_{|v|\geq R} f^{1-\alpha}\,|v|^{2(1-\alpha)}\,\frac{f^{\alpha}\,|\ln f|}{|v|^{2(1-\alpha)}}\,dv \\ &\leq C_2(\alpha)\left(\int_{|v|\geq R} f\,|v|^2\,dv\right)^{1-\alpha}\left(\int_{|v|\geq R}|v|^{-2(1-\alpha)/\alpha}\,dv\right)^{\alpha}.\end{aligned}$$

We fix $\alpha = 1/5$ and conclude that

$$\int_{|v|\geq R} f\,|\ln f|\,dv \leq C\,\frac{E_0^{1-\alpha}}{R}. \tag{3.5}$$

On the other hand, using (3.3) with $\varepsilon = 1/2$ leads to

$$\begin{aligned}\int_{|v|\geq R}(1-f)\,|\ln(1-f)|\,dv &\leq 2\int_{\{|v|\geq R\}\cap\{f\leq 1/2\}} f(1-f)\,dv + \frac{1}{e}\int_{\{|v|\geq R\}\cap\{f\geq 1/2\}} f\,dv \\ &\leq \frac{2}{R^2}\int f\,|v|^2\,dv + \frac{1}{eR^2}\int f\,|v|^2\,dv.\end{aligned}$$

Hence, for $R \geq 1$,

$$\int_{|v|\geq R}(1-f)\,|\ln(1-f)|\,dv \leq \frac{3E_0}{R^2} \leq \frac{3E_0}{R}. \tag{3.6}$$

From (3.5) and (3.6), we deduce

$$\int_{|v|\geq R}\big(f\,|\ln f| + (1-f)\,|\ln(1-f)|\big)\,dv \leq \frac{C_3(E_0)}{R}. \tag{3.7}$$

Step 3. Substituting the inequalities (3.4) and (3.7) into (3.1) gives

$$S_0 - \frac{C_1(1/5) + C_3(E_0)}{R} \leq 2R^{20}\int_{B_R} f(1-f)\,dv.$$

The choice

$$R_* = 2\,\frac{C_1(1/5) + C_3(E_0)}{S_0},$$

then completes the proof of Lemma 3.1. □

Proof of Proposition 2.3. Owing to Lemma 3.1, the remainder of the proof of Proposition 2.3 is similar to that of [10, Proposition 4], to which we refer. □

3.2 Propagation of moments

We now show (formally) the propagation of moments for solutions to the LFD equation (2.1), (2.2), which, in turn, implies an H^1-estimate (still formally). All the computations we perform here will be justified in Section 4.2 by means of smooth approximating solutions.

Let f be a smooth solution to (2.1), (2.2). Multiplying (2.1) by 1 and $|v|^2$ and integrating with respect to v lead, after some integrations by parts, to the conservation of mass (2.4) and energy (2.5). Also, after multiplying (2.1) by $\ln f - \ln(1-f)$ and integrating over

$\mathbb{R}^3$, the nonnegativity of the matrix a ensures that the entropy $S(f)$ is a non-decreasing function of time. From now on, C_i, $i \geq 1$ denote positive constants depending only on γ, M_{in}, E_{in} and S_{in}. The dependance of the C_i's upon additional parameters will be indicated explicitly.

Lemma 3.2 *Assume that $f_{in} \in L^1_{2s}(\mathbb{R}^3)$ for some $s > 1$. Then, for every $T > 0$, there exists a constant $\Gamma(s, T, \|f_{in}\|_{L^1_{2s}})$ depending only on s, T and $\|f_{in}\|_{L^1_{2s}}$ such that*

$$\sup_{t\in[0,T]} \|f(t)\|_{L^1_{2s}} + \int_0^T \iint |v - v_*|^\gamma \, |v_*|^{2s} \, f f_*(1 - f_*) \, dv \, dv_* \, d\tau \leq \Gamma(s, T, \|f_{in}\|_{L^1_{2s}}).$$

In particular, $f(1 - f) \in L^1_{loc}(\mathbb{R}_+; L^1_{2s+\gamma}(\mathbb{R}^3))$.

Proof. Let φ be a smooth function on $\mathbb{R}^3$ and multiply (2.1) by φ. After integrating over $\mathbb{R}^3$ and some integrations by parts, we obtain:

$$\frac{d}{dt} \int f(t, v) \, \varphi(v) \, dv = \sum_{i,j} \iint f f_*(1 - f_*) \, a_{i,j}(v - v_*) \, \partial^2_{i,j}\varphi \, dv \, dv_* + \iint f f_* \, [2 - f - f_*] \, b(v - v_*) \cdot \nabla\varphi \, dv \, dv_*. \quad (3.8)$$

We take $\varphi(v) = \Phi\big(|v|^2\big)$ in (3.8), where Φ is a convex function. As

$$\begin{aligned} \sum_i a_{i,i}(v - v_*) &= 2 \, |v - v_*|^{\gamma+2}, \\ \sum_{i,j} a_{i,j}(v - v_*) \, v_i v_j &= |v - v_*|^\gamma \, \big(|v|^2 |v_*|^2 - (v \cdot v_*)^2\big), \\ \sum_j b_j(v - v_*) \, v_j &= -2 \, |v - v_*|^\gamma \, \big(|v|^2 - (v \cdot v_*)\big), \end{aligned}$$

the formula (3.8) becomes

$$\frac{d}{dt} \int f(t, v) \, \Phi\big(|v|^2\big) \, dv = 4 \iint f f_*(1 - f_*) \, |v - v_*|^\gamma \, \Lambda^\Phi(v, v_*) \, dv \, dv_*,$$

where

$$\Lambda^\Phi(v, v_*) = \big(|v_*|^2 - (v \cdot v_*)\big)\big(\Phi'\big(|v|^2\big) - \Phi'\big(|v_*|^2\big)\big) + \big(|v|^2|v_*|^2 - (v \cdot v_*)^2\big)\Phi''\big(|v|^2\big).$$

Since Φ is convex, Φ'' is nonnegative and, consequently,

$$\Lambda^\Phi(v, v_*) \leq \big(|v_*|^2 - (v \cdot v_*)\big)\big(\Phi'\big(|v|^2\big) - \Phi'\big(|v_*|^2\big)\big) + |v|^2|v_*|^2\Phi''\big(|v|^2\big).$$

Let $\Phi(r) = r^s$, $s > 1$. Since $(v \cdot v_*) \leq |v||v_*|$, we deduce (with the notation $\Lambda^s = \Lambda^\Phi$) that

$$\Lambda^s(v, v_*) \leq s\Big[s \, |v|^{2s-2} \, |v_*|^2 - |v_*|^{2s} + |v||v_*|\big(|v|^{2s-2} + |v_*|^{2s-2}\big)\Big]. \quad (3.9)$$

As $s > 1$, we have $2s - 1 > 1$ and Young's inequality ensures that

$$x^{2s-2} \, y^2 = x^{2s-2} \, y^{(2s-2)/(2s-1)} \, y^{2s/(2s-1)} \leq \frac{2s - 2}{2s - 1} \, x^{2s-1} \, y + \frac{1}{2s - 1} \, y^{2s}.$$

Substituting this inequality for $x = |v|$, $y = |v_*|$ into (3.9) yields

$$\Lambda^s(v, v_*) \le s\Big[(s+1)\,|v|^{2s-1}\,|v_*| + |v||v_*|^{2s-1} - \frac{s-1}{2s-1}|v_*|^{2s}\Big].$$

Since $f \ge 0$ and $|v - v_*|^\gamma \le |v|^\gamma + |v_*|^\gamma$ $(\gamma \ge 0)$, we finally obtain

$$\frac{d}{dt}\int f(t,v)\,|v|^{2s}\,dv + 4\,s\,\frac{s-1}{2s-1}\iint |v-v_*|^\gamma\,|v_*|^{2s}\,ff_*(1-f_*)\,dv\,dv_*$$
$$\le 4s\iint ff_*\,(|v|^\gamma + |v_*|^\gamma)\Big[(s+1)\,|v|^{2s-1}\,|v_*| + |v||v_*|^{2s-1}\Big]\,dv\,dv_*.$$

Now,

$$\begin{aligned}(|v|^\gamma + |v_*|^\gamma)\Big[(s+1)\,|v|^{2s-1}\,|v_*| + |v||v_*|^{2s-1}\Big] \;\le\;& (s+1)\Big[|v|^{2s+\gamma-1}\,|v_*| + |v|^{2s-1}\,|v_*|^{1+\gamma}\Big]\\ &+\Big[|v|^{1+\gamma}|v_*|^{2s-1} + |v||v_*|^{2s+\gamma-1}\Big],\end{aligned}$$

and Young's inequality ensures that

$$\max\Big\{|v|, |v|^{\gamma+1}\Big\} \le 1 + |v|^2 \quad \text{and} \quad \max\Big\{|v|^{2s-1}, |v|^{2s+\gamma-1}\Big\} \le 1 + |v|^{2s}.$$

Therefore,

$$\frac{d}{dt}M_{2s}(f) + 4\,s\,\frac{s-1}{2s-1}\iint |v-v_*|^\gamma\,|v_*|^{2s}\,ff_*(1-f_*)\,dv\,dv_* \le C_1(s) + C_2(s)\,M_{2s}(f). \tag{3.10}$$

Thanks to the Gronwall lemma, we first conclude that, for every $T \ge 0$,

$$M_{2s}(f)(t) \le M_{2s}(f_{in}) + C_3(s,T), \qquad t \in [0,T]. \tag{3.11}$$

We next integrate (3.10) over $(0,T)$ and deduce from (3.11) that

$$\int_0^T \iint |v-v_*|^\gamma\,|v_*|^{2s}\,ff_*(1-f_*)\,dv\,dv_*\,d\tau \le (M_{2s}(f_{in}) + 1)\,C_4(s,T).$$

Since $|v - v_*|^\gamma \ge |v_*|^\gamma - |v|^\gamma$, we infer that

$$\|f_{in}\|_{L^1}\int_0^T\int f_*(1-f_*)|v_*|^{2s+\gamma}\,dv_* \le (M_{2s}(f_{in}) + 1)\,C_4(s,T) + \|f_{in}\|_{L^1_2}\,\|f\|_{L^\infty(0,T;L^1_{2s})}, \tag{3.12}$$

which completes the proof. □

Remark 3.3 *Unlike the classical Landau equation for which $M_{2s}(f)$ becomes instantaneously finite for positive times and $s > 1$, we obtain here the propagation of these moments but not their appearance. This is due to the term $f_*(1 - f_*)$ in (3.12). Consequently, we do not recover the same smoothness as in [10, Theorems 3 and 5].*

Lemma 3.4 *For every $T > 0$, there exists a constant $C(s,T)$ such that*

$$K\int_0^T\int |\nabla f|^2\,\big(1+|v|^2\big)^{s+\gamma/2}\,dv\,d\tau$$
$$\le C(s,T)\,\Big[1 + \|f(1-f)\|_{L^1(0,T;L^1_{2+\gamma})}\Big]\,\|f\|_{L^\infty(0,T;L^1_{2s+\gamma})} + \|f_{in}\|_{L^1_{2s}}. \tag{3.13}$$

Proof. Let $s \geq 0$ and $f_{in} \in L^1_{2s}(\mathbb{R}^3)$. Since $0 \leq f \leq 1$, (2.4), (2.5) and Lemma 3.2 imply that $f \in L^\infty\big(0,T; L^2_{2s}(\mathbb{R}^3)\big)$. It follows from (2.1) that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int f^2\big(1+|v|^2\big)^s dv &= -\int \Big(\overline{A}\nabla f - \overline{b} f(1-f)\Big)\nabla\Big(f\big(1+|v|^2\big)^s\Big)\, dv \\
&= -\int \overline{A}\nabla f \nabla f\big(1+|v|^2\big)^s dv - 2\,s\int \overline{A} f\nabla f\, v\big(1+|v|^2\big)^{s-1} dv \\
&+ \int f(1-f)\overline{b}\cdot\nabla f\big(1+|v|^2\big)^s dv + 2\,s\int \overline{b}\cdot v\, f^2(1-f)\big(1+|v|^2\big)^{s-1} dv. \quad (3.14)
\end{aligned}$$

On the one hand, since $S(f)$ is a non-decreasing function and f satisfies (2.5), Proposition 2.3 implies that

$$\int \overline{A}\nabla f\nabla f\big(1+|v|^2\big)^s dv \geq K\int |\nabla f|^2\,\big(1+|v|^2\big)^{s+\gamma/2}\, dv.$$

On the other hand, it is easy to see that there exists a constant C such that

$$\begin{aligned}
\Big|\nabla\cdot\Big(\overline{A}v\,\big(1+|v|^2\big)^{s-1}\Big)\Big| &\leq C\,\|f(1-f)\|_{L^1_{2+\gamma}}\,\big(1+|v|^2\big)^{s+\gamma/2}, \quad (3.15)\\
\big|\nabla\cdot\big(\overline{b}\big(1+|v|^2\big)^s\big)\big| &\leq C\,\|f\|_{L^1_2}\,\big(1+|v|^2\big)^{s+\gamma/2},\\
\big|\,\overline{b}\big| &\leq C\,\|f\|_{L^1_2}\,\big(1+|v|^2\big)^{(1+\gamma)/2},
\end{aligned}$$

so that

$$\int \overline{b}\cdot v\, f^2(1-f)\big(1+|v|^2\big)^{s-1} dv \leq C\,\|f\|_{L^1_2}\int f^2\big(1+|v|^2\big)^{s+\gamma/2} dv,$$

and

$$\begin{aligned}
\int f(1-f)\,\overline{b}\cdot\nabla f\big(1+|v|^2\big)^s dv &= -\int\Big(\frac{1}{2}f^2 - \frac{1}{3}f^3\Big)\,\nabla\cdot\Big(\overline{b}\big(1+|v|^2\big)^s\Big)dv\\
&\leq C\,\|f\|_{L^1_2}\int f^2\big(1+|v|^2\big)^{s+\gamma/2}dv,\\
-2\int \overline{A}f\nabla f\, v\,\big(1+|v|^2\big)^{s-1}dv &= \int f^2\,\nabla\cdot\Big[\overline{A}\,v\,\big(1+|v|^2\big)^{s-1}\Big]\,dv\\
&\leq C\,\|f(1-f)\|_{L^1_{2+\gamma}}\int f^2\big(1+|v|^2\big)^{s+\gamma/2}dv.
\end{aligned}$$

Substituting the previous estimates into (3.14) and using (2.4) and (2.5) yield (3.13) after integrating with respect to time. □

4 Existence

This section is devoted to the proof of the existence part of Theorem 2.2. First we investigate a regularized problem and show the existence and smoothness of a solution. Indeed, a first difficulty common to both the Landau and LFD equations lies in the fact that the coefficients of the elliptic operator $Q_L(f)$ are unbounded. We thus approximate them by bounded ones. However, the coefficients remain non-local, which is the second difficulty to be faced. The existence of approximated solutions follows from a fixed point method but, unlike the classical Landau equation, this method has to be applied to a nonlinear equation. Finally, we obtain solutions to the LFD equation as cluster points of sequences of approximated solutions. At this stage, owing to the cubic nature of the LFD equation, weak convergence is not sufficient.

4.1 The regularized problem

Let $(\Psi_\varepsilon)_{\varepsilon>0}$ be a family of smooth bounded functions on $\mathbb{R}_+$ which coincide with $\Psi(r) = r^{\gamma+2}$ for $0 < \varepsilon < r < \varepsilon^{-1}$ and enjoy the following properties:

- (i) The functions $\Psi'_\varepsilon, \Psi''_\varepsilon, \Psi^{(3)}_\varepsilon$ and $\Psi^{(4)}_\varepsilon$ are bounded;
- (ii) For $0 < r < \varepsilon^{-1}, \quad \Psi_\varepsilon(r) \geq r^{\gamma+2}/2$;
  For $r > \varepsilon^{-1}, \quad \Psi_\varepsilon(r) \geq \varepsilon^{-(\gamma+2)}/2 > 0$;
- (iii) For every $r > 0, \quad \Psi_\varepsilon(r) \leq r^2(1+r^\gamma) \quad$ and $\quad |\Psi'_\varepsilon(r)| \leq (\gamma+2)\, r\,(1+r^\gamma)$;
- (iv) For $0 < r < \varepsilon$, $\Psi_\varepsilon(r) = r^2\, \nu_\varepsilon(r)$, with $\nu_\varepsilon \in \mathcal{C}^\infty([0,\varepsilon])$, $\nu_\varepsilon(0) = 1$, $\nu'_\varepsilon(0) = 0$ and $\nu''_\varepsilon(0) = 0$.

For $(i,j) \in [\![1,3]\!]^2$, we set

$$a^\varepsilon(z) = \left(a^\varepsilon_{i,j}(z)\right)_{i,j} \qquad \text{with} \qquad a^\varepsilon_{i,j}(z) = \Psi_\varepsilon(|z|)\left(\delta_{i,j} - \frac{z_i z_j}{|z|^2}\right),$$

$$b^\varepsilon_i(z) = \sum_k \partial_k a^\varepsilon_{i,k}(z) = -\frac{2\, z_i}{|z|^2}\, \Psi_\varepsilon(|z|),$$

$$c^\varepsilon(z) = \sum_{k,l} \partial^2_{kl} a^\varepsilon_{k,l}(z) = -\frac{2}{|z|^2}\left[\Psi_\varepsilon(|z|) + |z|\, \Psi'_\varepsilon(|z|)\right],$$

and consider the regularized problem

$$\begin{cases} \partial_t f = \nabla\cdot\left(\overline{A}^{f,\varepsilon}\nabla f - \overline{b}^{f,\varepsilon} f(1-f)\right) + \varepsilon\Delta f, \\ f(0,.) = f_{in}. \end{cases} \tag{4.1}$$

We first note that, thanks to the properties of Ψ_ε, we have the following result.

Lemma 4.1 *The functions $a^\varepsilon_{i,j}$ and b^ε_i belong to $C^4_b(\mathbb{R}^3)$. The function c^ε belongs to $C^2_b(\mathbb{R}^3)$.*

We set

$$K_\varepsilon = \max_{i,j} \|a^\varepsilon_{i,j}\|_{C^4_b} + \max_i \|b^\varepsilon_i\|_{C^4_b} + \|c^\varepsilon\|_{C^2_b}.$$

We next investigate the well-posedness of (4.1).

Theorem 4.2 *Consider $f_{in} \in \mathcal{C}^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \cap W^{3,\infty}(\mathbb{R}^3)$ such that*

$$0 < \alpha_1\, e^{-\beta_1|v|^2} \leq f_{in}(v) \leq \frac{\alpha_2\, e^{-\beta_2|v|^2}}{1+\alpha_2\, e^{-\beta_2|v|^2}} < 1, \qquad \forall\, v \in \mathbb{R}^3, \tag{4.2}$$

for positive constants α_1, α_2, β_1 and β_2. Let $\varepsilon > 0$ and $T > 0$. Then, there exists a solution f^ε to the regularized problem (4.1) with initial condition f_{in} such that, for every $s > 0$, f^ε belongs to $L^\infty\big(0,T; L^1_{2s}(\mathbb{R}^3)\big) \cap L^2\big(0,T; H^1_{2s}(\mathbb{R}^3)\big)$.

Let $\beta'_1 \geq \beta_1$, D, E, F and C_L be five positive constants, the values of which we will specify later. We set

$$\mathcal{C} = \left\{ f \in \mathcal{C}\big([0,T]; L^1(\mathbb{R}^3)\big); \begin{array}{c} 0 \le f \le 1, \qquad \forall\, s,t \in [0,T], \quad \forall \varphi \in \mathcal{C}^2_b(\mathbb{R}^3), \\ \displaystyle\int f(t,v)\, dv = \int f_{in}(v)\, dv \\ \displaystyle\left| \int \big(f(t,v) - f(s,v)\big)\, \varphi(v)\, dv \right| \le C_L\, \|\varphi\|_{\mathcal{C}^2_b}\, |t-s| \\ \displaystyle\left| \int \big(f(1-f)(t,v) - f(1-f)(s,v)\big)\, \varphi(v)\, dv \right| \\ \le C_L\, \|\varphi\|_{\mathcal{C}^2_b}\, |t-s| \\ \displaystyle \alpha_1 e^{-\beta_1' |v|^2} e^{-Dt} \le f(t,v) \le \frac{\alpha_2\, e^{Et}\, e^{-\beta_2 |v|^2/(1+Ft)}}{1+\alpha_2\, e^{Et}\, e^{-\beta_2 |v|^2/(1+Ft)}} \end{array} \right\}.$$

For $g \in \mathcal{C}$, we consider the following quasilinear problem

$$\begin{cases} \partial_t f = \nabla \cdot \Big[\big(\overline{A}^{g,\varepsilon} + \varepsilon I_3\big) \nabla f - \overline{b}^{g,\varepsilon} f(1-f) \Big], \\ f(0,.) = f_{in}, \end{cases} \tag{4.3}$$

where I_3 denotes the identity matrix of $\mathbb{R}^3$.

The existence of solutions to (4.1) will follow from the existence of solutions to (4.3) by means of a fixed point method. We thus first study the latter and prove the following result.

Theorem 4.3 *Let $\delta \in (0,1)$ and $\varepsilon > 0$. For each $g \in \mathcal{C}$, there exists a unique classical solution $f^\varepsilon \in \mathcal{H}^{2+\delta,\,(2+\delta)/2}\big([0,T] \times \mathbb{R}^3\big)$ to (4.3) and there is a constant Λ depending only on f_{in}, δ, T, ε and C_L such that*

$$\|f^\varepsilon\|_{\mathcal{H}^{2+\delta,\,(2+\delta)/2}} \le \Lambda.$$

Moreover, there exist constants β_1', D, E, F and C_L depending only on f_{in}, T and ε such that f^ε belongs to $\mathcal{C}$.

For $T > 0$, $l > 0$, $l \notin \mathbb{N}$ and Ω a domain of $\mathbb{R}^3$, we consider Hölder spaces $\mathcal{H}^{l,\,l/2}([0,T] \times \Omega)$, whose norm is

$$\begin{aligned} \|f\|_{\mathcal{H}^{l,l/2}} = & \sup_{0 \le t \le T,\, v \in \mathbb{R}^3} \sum_{|\alpha|+2r \le [l]} |\partial_t^r \partial_\alpha f(t,v)| \\ & + \sup_{0 \le t \le T,\, v \ne w} \sum_{|\alpha|+2r = [l]} \frac{|\partial_t^r \partial_\alpha f(t,v) - \partial_t^r \partial_\alpha f(t,w)|}{|v-w|^{l-[l]}} \\ & + \sup_{s \ne t,\, v \in \mathbb{R}^3} \sum_{|\alpha|+2r = [l]} \frac{|\partial_t^r \partial_\alpha f(t,v) - \partial_t^r \partial_\alpha f(s,v)|}{|t-s|^{(l-[l])/2}}, \end{aligned}$$

where $[l]$ denotes the integer part of l and $\alpha \in \mathbb{N}^3$.

Thanks to Lemma 4.1 and to the properties of $\mathcal{C}$, the coefficients of the parabolic operator in (4.3) have the following regularity properties:

Lemma 4.4 *Let $\varepsilon > 0$, $\delta \in (0,1)$ and $g \in \mathcal{C}$. For every $(i,j,k) \in [\![1,3]\!]^3$, the functions $\overline{A}^{g,\varepsilon}_{i,j}$, $\overline{b}^{g,\varepsilon}_i$, $\partial_k \overline{A}^{g,\varepsilon}_{i,j}$ and $\overline{c}^{g,\varepsilon}$ belong to the Hölder space $\mathcal{H}^{\delta,\,\delta/2}\big([0,T] \times \mathbb{R}^3\big)$, with*

$$\max_{i,j} \|\overline{A}^{g,\varepsilon}_{i,j}\|_{W^{1,\infty}} + \max_i \|\overline{b}^{g,\varepsilon}_i\|_{L^\infty} + \|\overline{c}^{g,\varepsilon}\|_{L^\infty} \le K_\varepsilon \|f_{in}\|_{L^1}. \tag{4.4}$$

Moreover, for every bounded domain Ω *of* $\mathbb{R}^3$*, the functions* $\overline{A}^{g,\varepsilon}_{i,j}$, $\overline{b}^{g,\varepsilon}_i$, $\partial_k \overline{A}^{g,\varepsilon}_{i,j}$ *and* $\partial_k \overline{b}^{g,\varepsilon}_i$ *belong to the Hölder space* $\mathcal{H}^{1+\delta,\,(1+\delta)/2}\big([0,T]\times\Omega\big)$.

Proof of Theorem 4.3. Owing to the uniform ellipticity

$$\forall v \in \mathbb{R}^3,\ \forall \xi \in \mathbb{R}^3, \qquad \varepsilon\,|\xi|^2 \le \sum_{i,j} (\overline{A}^{g,\varepsilon}_{i,j}(v) + \varepsilon\delta_{i,j})\,\xi_i\xi_j \le (\,3\,K_\varepsilon \|f_{in}\|_{L^1} + \varepsilon\,)\,|\xi|^2, \tag{4.5}$$

and classical arguments, the maximum principle and [17, Theorem 5.8.1] imply the existence and uniqueness of a solution $f^\varepsilon \in \mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$ to (4.3). This solution satisfies

$$0 \le f^\varepsilon(t,v) \le 1, \tag{4.6}$$

and there exists a constant Λ depending only on f_{in}, δ, T, ε and C_L such that

$$\|f^\varepsilon\|_{\mathcal{H}^{2+\delta,(2+\delta)/2}} \le \Lambda,$$

(see the Appendix for a sketch of proof).

We next show that we can choose constants β_1', D, E, F and C_L such that $f^\varepsilon \in \mathcal{C}$. First, we verify that, for every $(t,v) \in [0,T]\times\mathbb{R}^3$,

$$\alpha_1 e^{-\beta_1'|v|^2} e^{-Dt} \le f^\varepsilon(t,v) \le \frac{\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}}{1+\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}}. \tag{4.7}$$

Indeed, introducing

$$\varphi_{inf}(t,v) = \alpha_1\, e^{-\beta_1'|v|^2}\, e^{-Dt},$$

and the parabolic operator $\mathcal{L}$ defined by

$$\mathcal{L}u = \partial_t u - \sum_{i,j} \left(\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon\,\delta_{i,j}\right) \partial^2_{i,j} u - \sum_i \left[\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i (1-2f^\varepsilon)\right]\partial_i u + \overline{c}^{g,\varepsilon}\,(1-f^\varepsilon)\,u,$$

we see that $\mathcal{L}\varphi_{inf} \le 0$ as soon as

$$\beta_1' \ge \frac{1}{4\varepsilon} \quad \text{ and } \quad D \ge 12\beta_1' K_\varepsilon^2 \|f_{in}\|^2_{L^1} + 6\beta_1'(K_\varepsilon\|f_{in}\|_{L^1} + \varepsilon) + K_\varepsilon\|f_{in}\|_{L^1},$$

whence

$$\alpha_1 e^{-\beta_1'|v|^2} e^{-Dt} \le f^\varepsilon(t,v), \qquad \forall (t,v) \in [0,T]\times\mathbb{R}^3,$$

by the comparison principle [17, Theorem 1.2.1].

We next set

$$\varphi_{sup}(t,v) = \frac{\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}}{1+\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}},$$

and let $\mathcal{M}$ be the semilinear operator defined by

$$\mathcal{M}u = \partial_t u - \sum_{i,j} \left(\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon\,\delta_{i,j}\right) \partial^2_{i,j} u - \sum_i \left[\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i (1-2u)\right]\partial_i u + \overline{c}^{g,\varepsilon}(1-u)\,u.$$

For

$$E \ge 12K_\varepsilon^2\,\|f_{in}\|_{L^1} + K_\varepsilon\,\|f_{in}\|^2_{L^1} \quad \text{ and } \quad F \ge 12\,\beta_2 K_\varepsilon \|f_{in}\|_{L^1} + 4\,\beta_2\,\varepsilon + \beta_2,$$

we have $\mathcal{M}\varphi_{sup} \geq 0 = \mathcal{M}f^\varepsilon$. Owing to the regularity of the coefficients of the parabolic operator, we are in a position to apply the comparison principle [19, Theorem 9.1] to obtain that

$$f^\varepsilon(t,v) \leq \frac{\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}}{1+\alpha_2\, e^{Et}\, e^{-\beta_2\,|v|^2/(1+Ft)}}.$$

It next readily follows from (4.7) and the continuity of f^ε that $f^\varepsilon \in \mathcal{C}([0,T];L^1(\mathbb{R}^3))$. In addition, classical truncation arguments, (4.4) and (4.7) allow us to check that

$$\forall t \in [0,T], \qquad \int f^\varepsilon(t,v)\,dv = \int f_{in}(v)\,dv. \tag{4.8}$$

It remains now to verify the two Lipschitz properties and this will be the aim of the three following lemmas. We only give formal calculations but they can be rigorously justified by standard truncation arguments.

Lemma 4.5 *For every $r \geq 0$, f^ε belongs to $L^2(0,T;H^1_{2r}(\mathbb{R}^3))$. Moreover, there exists a constant C depending only on f_{in}, r, ε and T such that,*

$$\|f^\varepsilon\|_{L^2(0,T;H^1_{2r})} \leq C.$$

Proof. Let $r \geq 0$. We multiply (4.1) by $f^\varepsilon\big(1+|v|^2\big)^r$ and we integrate with respect to v to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |f^\varepsilon|^2\,(t,v)\,\big(1+|v|^2\big)^r\,dv \\
&= -\int (\overline{A}^{g,\varepsilon}+\varepsilon I_3)\nabla f^\varepsilon \nabla f^\varepsilon\,\big(1+|v|^2\big)^r\,dv - 2r\int (\overline{A}^{g,\varepsilon}+\varepsilon I_3)\nabla f^\varepsilon\, v\, f^\varepsilon\,\big(1+|v|^2\big)^{r-1}\,dv \\
&\quad + \int f^\varepsilon(1-f^\varepsilon)\,\overline{b}^{g,\varepsilon}\cdot\nabla f^\varepsilon\,\big(1+|v|^2\big)^r\,dv + 2r\int (f^\varepsilon)^2\,(1-f^\varepsilon)\,\overline{b}^{g,\varepsilon}\cdot v\,\big(1+|v|^2\big)^{r-1}\,dv.
\end{aligned}$$

After integrating over $(0,t)$, we infer from (4.4), (4.5) and Young's inequality that

$$\begin{aligned}
&\frac{1}{2}\int |f^\varepsilon|^2\,(t,v)\,\big(1+|v|^2\big)^r\,dv + \varepsilon\int_0^t\int |\nabla f^\varepsilon|^2\,\big(1+|v|^2\big)^r\,dv\,d\tau \\
&\leq \frac{\varepsilon}{3}\int_0^t\int |\nabla f^\varepsilon|^2\,\big(1+|v|^2\big)^r\,dv\,d\tau + C_\varepsilon[3\,K_\varepsilon\|f_{in}\|_{L^1}+\varepsilon]^2\int_0^t\int f^\varepsilon\,\big(1+|v|^2\big)^{r-1}\,dv\,d\tau \\
&\quad + \frac{\varepsilon}{3}\int_0^t\int |\nabla f^\varepsilon|^2\,\big(1+|v|^2\big)^r\,dv\,d\tau + C_\varepsilon K_\varepsilon^2\|f_{in}\|^2_{L^1}\int_0^t\int f^\varepsilon\,\big(1+|v|^2\big)^r\,dv\,d\tau \\
&\quad + CK_\varepsilon\|f_{in}\|_{L^1}\int_0^t\int f^\varepsilon\,\big(1+|v|^2\big)^r\,dv\,d\tau + \frac{1}{2}\int |f_{in}|^2\,\big(1+|v|^2\big)^r\,dv.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varepsilon\int_0^t\int |\nabla f^\varepsilon|^2\,\big(1+|v|^2\big)^r\,dv\,d\tau \quad &\leq \quad C(\varepsilon,M_{in})\int_0^t\int f^\varepsilon\,\big(1+|v|^2\big)^r\,dv\,d\tau \\
&\qquad + \frac{3}{2}\int |f_{in}|^2\,\big(1+|v|^2\big)^r\,dv,
\end{aligned}$$

and (4.7) implies that the right-hand side of the above inequality is bounded. □

Lemma 4.6 *The function f^ε belongs to $L^\infty\big(0,T;H^1\big(\mathbb{R}^3\big)\big)$. Moreover, there exists a constant G depending only on f_{in}, ε and T such that,*

$$\|f^\varepsilon\|_{L^\infty(0,T;H^1)} \le G.$$

Proof. We first observe that

$$f^\varepsilon \in \mathcal{H}^{2+\delta,(2+\delta)/2}([0,T]\times\mathbb{R}^3)\cap\mathcal{H}^{3+\delta,\,(3+\delta)/2}([0,T]\times\Omega)$$

for each bounded domain $\Omega\subset\mathbb{R}^3$ by [17, Theorem 5.8.1]. We may thus differentiate (4.3) with respect to v_k and obtain

$$\partial_t\partial_k f^\varepsilon = \nabla\cdot\Big[(\overline{A}^{g,\varepsilon}+\varepsilon I_3)\nabla\partial_k f^\varepsilon + \partial_k\overline{A}^{g,\varepsilon}\nabla f^\varepsilon - \overline{b}^{g,\varepsilon}(1-2f^\varepsilon)\partial_k f^\varepsilon - \partial_k\overline{b}^{g,\varepsilon} f^\varepsilon(1-f^\varepsilon)\Big].$$

Hence,

$$\begin{aligned}\frac12\frac{d}{dt}\int(\partial_k f^\varepsilon)^2\,dv \;=\; & -\int(\overline{A}^{g,\varepsilon}+\varepsilon I_3)\,\nabla\partial_k f^\varepsilon\,\nabla\partial_k f^\varepsilon\,dv - \int \partial_k\overline{A}^{g,\varepsilon}\,\nabla f^\varepsilon\,\nabla\partial_k f^\varepsilon\,dv\\ & + \int(1-2f^\varepsilon)\,\overline{b}^{g,\varepsilon}\cdot\nabla\partial_k f^\varepsilon\,\partial_k f^\varepsilon\,dv + \int f^\varepsilon(1-f^\varepsilon)\,\partial_k\overline{b}^{g,\varepsilon}\cdot\nabla\partial_k f^\varepsilon\,dv,\end{aligned}$$

and (4.4), (4.5) and Young's inequality lead to

$$\begin{aligned}\int|\partial_k f^\varepsilon|^2\,dv + 2\varepsilon\int_0^t\int|\nabla\partial_k f^\varepsilon|^2\,dv\,d\tau \le \frac{3\varepsilon}{2}\int_0^t\int|\nabla\partial_k f^\varepsilon|^2\,dv\,d\tau\\ + C_\varepsilon K_\varepsilon^2\,\|f_{in}\|_{L^1}^2\,\|\nabla f^\varepsilon\|^2_{L^2(0,T;\,L^2)} + C_\varepsilon T\,K_\varepsilon^2\,\|f_{in}\|^3_{L^1} + \|f_{in}\|^2_{H^1}.\end{aligned}$$

Lemma 4.6 then readily follows from the above inequality by Lemma 4.5. □

Lemma 4.7 *There exists a constant C_L depending only on f_{in}, ε and T such that, for all $\varphi\in\mathcal{C}_b^2(\mathbb{R}^3)$ and $\sigma,t\in[0,T]$,*

$$\begin{aligned}\left|\int\big(f^\varepsilon(t,v)-f^\varepsilon(\sigma,v)\big)\,\varphi(v)\,dv\right| &\le C_L\,\|\varphi\|_{\mathcal{C}_b^2}\,|t-\sigma|,\\ \left|\int\big(f^\varepsilon(1-f^\varepsilon)(t,v)-f^\varepsilon(1-f^\varepsilon)(\sigma,v)\big)\,\varphi(v)\,dv\right| &\le C_L\,\|\varphi\|_{\mathcal{C}_b^2}\,|t-\sigma|.\end{aligned}$$

Proof. Let $\varphi\in\mathcal{C}_b^2(\mathbb{R}^3)$. Classical truncation arguments ensure that

$$\begin{aligned}\int f^\varepsilon(t,v)\,\varphi(v)\,dv - \int f^\varepsilon(\sigma,v)\,\varphi(v)\,dv = \int_\sigma^t d\tau\Big\{\sum_{i,j}\int\overline{A}^{g,\varepsilon}_{i,j}\,f^\varepsilon\,\partial^2_{i,j}\varphi\,dv + \int f^\varepsilon\,\overline{B}^{g,\varepsilon}\cdot\nabla\varphi\,dv\\ + \int f^\varepsilon\big(1-f^\varepsilon\big)\,\overline{b}^{g,\varepsilon}\cdot\nabla\varphi\,dv + \varepsilon\int f^\varepsilon\,\Delta\varphi\,dv\Big\},\end{aligned}\quad(4.9)$$

The first inequality of Lemma 4.7 then readily follows from (4.8), (4.9) and Lemma 4.4 with

$$C_L\ge C_1 = C\,(K_\varepsilon\|f_{in}\|_{L^1}+\varepsilon)\,\|f_{in}\|_{L^1}.$$

Similarly, we infer from (4.3) that, for $\varphi\in\mathcal{C}_b^2(\mathbb{R}^3)$, we have

$$\begin{aligned}&\int f^\varepsilon(1-f^\varepsilon)(t,v)\,\varphi(v)\,dv - \int f^\varepsilon(1-f^\varepsilon)(\sigma,v)\,\varphi(v)\,dv\\ &\quad= \int_\sigma^t d\tau\Big\{-\int(\overline{A}^{g,\varepsilon}+\varepsilon I_3)\,\nabla f^\varepsilon\,\Big[\big(1-2f^\varepsilon\big)\nabla\varphi - 2\,\varphi\nabla f^\varepsilon\Big]\,dv\\ &\quad\; - 2\int f^\varepsilon\big(1-f^\varepsilon\big)\,\varphi\,\overline{b}^{g,\varepsilon}\cdot\nabla f^\varepsilon\,dv + \int f^\varepsilon\big(1-f^\varepsilon\big)\big(1-2\,f^\varepsilon\big)\,\overline{b}^{g,\varepsilon}\cdot\nabla\varphi\,dv\Big\}.\end{aligned}\quad(4.10)$$

With the notation $M_1 : M_2 = \sum_{i,j} M_{1i,j} M_{2i,j}$ for any two matrices M_1 and M_2, we have

$$\int (\overline{A}^{g,\varepsilon} + \varepsilon I_3)\, \nabla f^\varepsilon \,(1-2f^\varepsilon) \nabla\varphi \, dv = -\int f^\varepsilon \big(1-f^\varepsilon\big)\, \nabla \cdot \Big((\overline{A}^{g,\varepsilon} + \varepsilon I_3)\, \nabla\varphi \Big)\, dv$$
$$= -\int f^\varepsilon \big(1-f^\varepsilon\big)\, \overline{B}^{g,\varepsilon} \cdot \nabla\varphi \, dv - \int f^\varepsilon \big(1-f^\varepsilon\big)\, (\overline{A}^{g,\varepsilon} + \varepsilon I_3) : \nabla^2\varphi \, dv,$$

and the identity (4.10) becomes

$$\int f^\varepsilon(1-f^\varepsilon)(t,v)\,\varphi(v)\,dv - \int f^\varepsilon(1-f^\varepsilon)(\sigma,v)\,\varphi(v)\,dv = \int_\sigma^t d\tau \Big\{ \int f^\varepsilon\big(1-f^\varepsilon\big)\, \overline{B}^{g,\varepsilon} \cdot \nabla\varphi \, dv$$
$$+ \int f^\varepsilon\big(1-f^\varepsilon\big)\, (\overline{A}^{g,\varepsilon} + \varepsilon I_3) : \nabla^2\varphi \, dv + 2\int (\overline{A}^{g,\varepsilon} + \varepsilon I_3)\, \nabla f^\varepsilon \, \nabla f^\varepsilon \, \varphi \, dv$$
$$- 2\int f^\varepsilon\big(1-f^\varepsilon\big)\, \varphi\, \overline{b}^{g,\varepsilon} \cdot \nabla f^\varepsilon \, dv + \int f^\varepsilon\big(1-f^\varepsilon\big)\, \big(1-2\, f^\varepsilon\big)\, \overline{b}^{g,\varepsilon} \cdot \nabla\varphi \, dv \Big\}.$$

The second inequality of Lemma 4.7 then follows with the help of (4.8) and Lemmas 4.4 and 4.6 with

$$C_L \geq C_2 = C\,(K_\varepsilon \|f_{in}\|_{L^1} + \varepsilon)\,(\|f_{in}\|_{L^1} + G^2).$$

Choosing $C_L = \max(C_1, C_2)$ completes the proof of Lemma 4.7. □

We have thus found β_1', D, E, F and C_L depending only on f_{in}, T and ε such that, if $g \in \mathcal{C}$, $f^\varepsilon \in \mathcal{C}$ and the proof of Theorem 4.3 is complete. □

Proof of Theorem 4.2. We fix β_1', D, E, F and C_L as in Theorem 4.3. For $g \in \mathcal{C}$, we denote by $\Phi(g)$ the unique solution $f^\varepsilon \in \mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$ to (4.3). Then $\Phi(g) \in \mathcal{C}$ by Theorem 4.3 and we now check that $\Phi : \mathcal{C} \to \mathcal{C}$ is continuous and compact for the topology of $\mathcal{C}\big([0,T];L^1(\mathbb{R}^3)\big)$.

Continuity of Φ. Consider $g_1 \in \mathcal{C}$, $g_2 \in \mathcal{C}$ and put $f_i^\varepsilon = \Phi(g_i)$ for $i = 1,2$. Then, $u = f_1^\varepsilon - f_2^\varepsilon$ satisfies

$$\partial_t u - \sum_{i,j} (\overline{A}_{i,j}^{g_1,\varepsilon} + \varepsilon\delta_{i,j})\partial_{i,j}^2 u - \sum_j \Big[\overline{B}_j^{g_1,\varepsilon} - \overline{b}_j^{g_1,\varepsilon}(1 - f_1^\varepsilon - f_2^\varepsilon)\Big]\partial_j u$$
$$+ \Big[\overline{c}^{g_1,\varepsilon}(1 - f_1^\varepsilon - f_2^\varepsilon) - \sum_j \overline{b}_j^{g_1,\varepsilon}(\partial_j f_1^\varepsilon + \partial_j f_2^\varepsilon)\Big] u = \Upsilon,$$

where

$$\Upsilon = \sum_{i,j} \Big(\overline{A}_{i,j}^{g_1,\varepsilon} - \overline{A}_{i,j}^{g_2,\varepsilon}\Big)\partial_{i,j}^2 f_2^\varepsilon + \sum_j \Big(\overline{B}_j^{g_1,\varepsilon} - \overline{B}_j^{g_2,\varepsilon}\Big)\partial_j f_2^\varepsilon$$
$$- \sum_j \Big(\overline{b}_j^{g_1,\varepsilon} - \overline{b}_j^{g_2,\varepsilon}\Big)(1 - 2f_2^\varepsilon)\partial_j f_2^\varepsilon - \Big(\overline{c}^{g_1,\varepsilon} - \overline{c}^{g_2,\varepsilon}\Big) f_2^\varepsilon(1 - f_2^\varepsilon).$$

Since u belongs to $\mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T]\times\mathbb{R}^3)$ and is bounded ($|u| \leq 1$), we infer from the maximum principle [17, Theorem 1.2.5] that

$$\sup_{[0,T]\times\mathbb{R}^3} \big|f_1^\varepsilon - f_2^\varepsilon\big| \leq \Big(\sup_{\mathbb{R}^3} \big|f_1^\varepsilon - f_2^\varepsilon\big|(0) + T \max_{[0,T]\times\mathbb{R}^3} |\Upsilon| \Big)\, e^{\omega T},$$

with

$$\omega = K_\varepsilon \, \|f_{in}\|_{L^1}(1+6\Lambda),$$

where the constant Λ is given by Theorem 4.3. Since $f_1^\varepsilon(0,.) = f_{in} = f_2^\varepsilon(0,.)$ and

$$|\Upsilon| \le K_\varepsilon \big|g_1 - g_2\big|_{\mathcal{C}([0,T];L^1)} \Big(\sum_{i,j} \sup |\partial^2_{i,j} f_2^\varepsilon| + 4 \sum_j \sup |\partial_j f_2^\varepsilon| + 1 \Big),$$

we deduce that

$$\sup_{[0,T]\times\mathbb{R}^3} \big|f_1^\varepsilon - f_2^\varepsilon\big| \le C T e^{\omega T} \big|g_1 - g_2\big|_{\mathcal{C}([0,T];L^1)}.$$

Now, for $R > 0$, we have

$$\begin{aligned} \big|f_1^\varepsilon - f_2^\varepsilon\big|_{\mathcal{C}([0,T];L^1)} &\le \sup_{t\in[0,T]} \int_{|v|\le R} |f_1^\varepsilon - f_2^\varepsilon|(t,v)\, dv + \sup_{t\in[0,T]} \int_{|v|\ge R} \big(|f_1^\varepsilon| + |f_2^\varepsilon|\big)(t,v)\, dv \\ &\le C\, R^3\, T\, e^{\omega T} \big|g_1 - g_2\big|_{\mathcal{C}([0,T];L^1)} + 2\, \alpha_2\, e^{ET} \int_{|v|\ge R} e^{-\beta_2 |v|^2/(1+FT)}\, dv \\ &\le C(T) R^3 \big|g_1 - g_2\big|_{\mathcal{C}([0,T];L^1)} + \frac{C(T)}{R^3}, \end{aligned}$$

whence

$$\big|f_1^\varepsilon - f_2^\varepsilon\big|_{\mathcal{C}([0,T];L^1)} \le C(T) \big|g_1 - g_2\big|^{1/2}_{\mathcal{C}([0,T];L^1)},$$

with the choice

$$R = \big|g_1 - g_2\big|^{-1/6}_{\mathcal{C}([0,T];L^1)}.$$

Compactness of Φ. For $m \ge 4$, we have $L^1_2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3) \subset L^1(\mathbb{R}^3) \subset (H^m(\mathbb{R}^3))'$ with a compact embedding $L^1_2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3) \subset L^1(\mathbb{R}^3)$. Since,

$$\begin{aligned} &\Phi(\mathcal{C}) \text{ is bounded in } L^\infty(0,T; L^1_2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)) \\ \text{and } \; &\partial_t \Phi(\mathcal{C}) \text{ is bounded in } L^r(0,T; (H^m(\mathbb{R}^3))'), \qquad \text{with } r > 1, \end{aligned}$$

by Theorem 4.3, we deduce from [27, Corollary 4], that $\Phi(\mathcal{C})$ is relatively compact in $\mathcal{C}([0,T]; L^1(\mathbb{R}^3))$.

We are now in a position to complete the proof of Theorem 4.2. Indeed, $\mathcal{C}$ is a non-empty, convex, closed and bounded subset from the Banach space $\mathcal{C}\big([0,T]; L^1\big(\mathbb{R}^3\big)\big)$. Since Φ is a compact and continuous map from $\mathcal{C}$ into $\mathcal{C}$, the Schauder fixed point theorem ensures the existence of a fixed point of Φ, that is, of a solution to (4.1). In addition, (4.7) and Lemma 4.5 warrant that f^ε has the desired properties. □

4.2 Uniform estimates

In order to pass to the limit as $\varepsilon \to 0$ in (4.3) and obtain a solution to (2.1), (2.2), we first need to establish uniform estimates on f^ε which do not depend on ε. These estimates are actually similar to those listed in Section 3. In the following, we denote by C any constant depending only on γ, M_{in}, E_{in} and S_{in}.

Lemma 4.8 *For all $\sigma, t \in [0,T]$, $\sigma \le t$, the function f^ε satisfies*

$$\int f^\varepsilon(t,v) dv = M_{in}, \tag{4.11}$$

$$\int f^\varepsilon(t,v)|v|^2 dv = E_{in} + 6\varepsilon M_{in} t \le E_{in} + 6\varepsilon M_{in} T, \tag{4.12}$$

$$S_{in} \le S(f^\varepsilon)(\sigma) \le S(f^\varepsilon)(t), \tag{4.13}$$

where $M_{in} = M_0(f_{in})$*,* $E_{in} = M_2(f_{in})$ *and* $S_{in} = S(f_{in})$*.*

Proof. Since $f^\varepsilon \in \mathcal{C}$, the first equality holds true. It next follows from (4.4), (4.7) and (4.9) with $\varphi(v) = |v|^2$ that

$$\int f^\varepsilon(t,v)\,|v|^2\,dv - \int f_{in}\,|v|^2\,dv = 6\,\varepsilon \int_0^t \int f^\varepsilon\,dv\,d\tau,$$

whence (4.12) by (4.11).

Finally, since f^ε is differentiable with respect to time and satisfies $0 < f^\varepsilon(t,v) < 1$ for every $(t,v) \in [0,T] \times \mathbb{R}^3$, by (4.7), we have

$$\partial_t \Big[f^\varepsilon \ln f^\varepsilon + (1-f^\varepsilon)\ln(1-f^\varepsilon) \Big] = \Big[\ln f^\varepsilon - \ln(1-f^\varepsilon) \Big] \partial_t f^\varepsilon.$$

Therefore, thanks to (4.1),

$$\begin{aligned} S(f^\varepsilon)(t) &= S(f_{in}) + \int_0^t d\tau \int \Big[\overline{A}^\varepsilon \nabla f^\varepsilon - \overline{b}^\varepsilon f^\varepsilon (1-f^\varepsilon) \Big] \frac{\nabla f^\varepsilon}{f^\varepsilon(1-f^\varepsilon)}\,dv \\ &\quad + \varepsilon \int_0^t d\tau \int \frac{|\nabla f^\varepsilon|^2}{f^\varepsilon(1-f^\varepsilon)}\,dv \\ &= S(f_{in}) + \frac{1}{2}\int_0^t d\tau \iint a^\varepsilon(v-v_*) \Big(f^\varepsilon_*(1-f^\varepsilon_*)\nabla f^\varepsilon - f^\varepsilon(1-f^\varepsilon)\nabla f^\varepsilon_* \Big) \\ &\quad \times \left(\frac{\nabla f^\varepsilon}{f^\varepsilon(1-f^\varepsilon)} - \frac{\nabla f^\varepsilon_*}{f^\varepsilon_*(1-f^\varepsilon_*)} \right) dv_*\,dv + \varepsilon \int_0^t d\tau \int \frac{|\nabla f^\varepsilon|^2}{f^\varepsilon(1-f^\varepsilon)}\,dv. \end{aligned}$$

Since the matrix a^ε is nonnegative, we conclude that the function $S(f^\varepsilon)$ is non-decreasing and (4.13) follows. □

We next consider the ellipticity of the diffusion matrix, the propagation of moments and the smoothness of f^ε. Proceeding as in the proof of Proposition 2.3 with the help of the properties of Ψ_ε, we first have the following results.

Proposition 4.9 *Denote by* R_* *the constant given by Lemma 3.1. For every* $0 < \varepsilon \le (3R_*)^{-1}$*, we have*

(i) Let $f \in \mathcal{Y}(E_{in}, S_{in})$*. Then there exists a constant* $K > 0$ *depending only on* γ*,* E_{in} *and* S_{in}*, such that, for every* $v \in \mathbb{R}^3$*,*

$$\forall\,\xi \in \mathbb{R}^3, \quad \sum_{i,j} \left(\overline{A}^\varepsilon_{i,j}(v) + \varepsilon\,\delta_{i,j} \right) \xi_i\,\xi_j \ge K \left(1+|v|^2\right)^{\gamma/2} \left[\min\left((\varepsilon|v|)^{-1}, 1/2 \right) \right]^{\gamma+2} |\xi|^2.$$

(ii) If $f(1-f) \in L^1_{\gamma+2}(\mathbb{R}^3)$*, then there exists a constant* $C > 0$ *depending only on* $M_{\gamma+2}(f)$ *and* $M_0(f)$ *such that, for every* $v \in \mathbb{R}^3$*,*

$$\forall \xi \in \mathbb{R}^3, \quad 0 \le \sum_{i,j} \left(\overline{A}^\varepsilon_{i,j}(v) + \varepsilon\,\delta_{i,j} \right) \xi_i\,\xi_j \le \left(C\left(1+|v|^{\gamma+2}\right) + \varepsilon \right) |\xi|^2.$$

In fact, the proof of the first point also gives a uniform (with respect to ε) ellipticity estimate.

Corollary 4.10 *For* $0 < \varepsilon \le (3R_*)^{-1}$*, there exists a constant* κ *depending only on* γ*,* E_{in} *and* S_{in}*, such that, for every* $f \in \mathcal{Y}(E_{in}, S_{in})$*,*

$$\forall\,\xi \in \mathbb{R}^3, \forall\,v \in \mathbb{R}^3, \quad \sum_{i,j} \left(\overline{A}^\varepsilon_{i,j}(v) + \varepsilon\,\delta_{i,j} \right) \xi_i\,\xi_j \ge \kappa\,\frac{|\xi|^2}{1+|v|^2}.$$

We next proceed as in the proof of Lemma 3.2 to show the following result.

Lemma 4.11 *For all $T > 0$, $s > 1$, there exists a constant Γ depending only on s, T and $\|f_{in}\|_{L^1_{2s}}$ such that*

$$\sup_{t\in[0,T]} \|f^\varepsilon(t)\|_{L^1_{2s}} + \int_0^T \iint \frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\, |v_*|^{2s}\, f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\, dv\, dv_*\, d\tau \le \Gamma(\|f_{in}\|_{L^1_{2s}}). \quad (4.14)$$

Remark 4.12 *The constant Γ increases with $\|f_{in}\|_{L^1_{2s}}$.*

Finally, a proof similar to that of Lemma 3.4 leads to the following H^1-estimate.

Lemma 4.13 *For all $T > 0$, $\varepsilon \in (0,1)$, $s \ge 0$, there exists a constant $C > 0$ depending only on s and T such that*

$$\begin{aligned} K \int_0^T \int |\nabla f^\varepsilon|^2 \left(1+|v|^2\right)^{s+\gamma/2} \left[\min\left((\varepsilon|v|)^{-1}, 1/2\right)\right]^{\gamma+2} dv\, d\tau \\ \le C \int_0^T \iint f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\, \frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\, |v_*|^2 \left(1+|v|^2\right)^{s-1} dv\, dv_*\, d\tau \\ +\ C\,(1+\varepsilon)\, \|f^\varepsilon\|_{L^\infty(0,T;L^1_{2s+\gamma})} \ +\ \|f_{in}\|_{L^1_{2s}}. \quad (4.15) \end{aligned}$$

In particular, for $s \in [0,1]$, we have, for every $\delta > 0$,

$$\begin{aligned} K \int_0^T \int |\nabla f^\varepsilon|^2 \left(1+|v|^2\right)^{s+\gamma/2} \left[\min\left((\varepsilon|v|)^{-1}, 1/2\right)\right]^{\gamma+2} dv\, d\tau \\ \le C\,\Gamma(\|f_{in}\|_{L^1_{2+\delta}}) + C\,(1+\varepsilon)\, \|f^\varepsilon\|_{L^\infty(0,T;L^1_{2s+\gamma})} + \|f_{in}\|_{L^1_{2s}}. \quad (4.16) \end{aligned}$$

Using Corollary 4.10 instead of Proposition 4.9 in the proof of Lemma 4.13 yields

Corollary 4.14 *For all $T > 0$, $\varepsilon \in (0,1)$, $s \ge 0$, there exists a constant $C > 0$ depending only on s and T such that*

$$\begin{aligned} K \int_0^T \int |\nabla f^\varepsilon|^2 \left(1+|v|^2\right)^{s-1} dv\, d\tau \\ \le C \int_0^T \iint f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\, \frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\, |v_*|^2 \left(1+|v|^2\right)^{s-1} dv\, dv_*\, d\tau \\ + C\,(1+\varepsilon) \|f^\varepsilon\|_{L^\infty(0,T;L^1_{2s+\gamma})} + \|f_{in}\|_{L^1_{2s}}. \end{aligned}$$

Proof of Lemma 4.13. A slight change to the proof of Lemma 3.4 is required here since we do not have an estimate on $f^\varepsilon(1-f^\varepsilon)$ in $L^1(0,T;L^1_{2+\gamma}(\mathbb{R}^3))$ because Ψ_ε is bounded. Thus, (3.15) has to be replaced by

$$\begin{aligned} \left| \int (f^\varepsilon)^2\, \nabla \cdot \left(\overline{A}^\varepsilon v \left(1+|v|^2\right)^{s-1} \right) dv \right| \\ \le C \iint f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\, \frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\, |v_*|^2 \left(1+|v|^2\right)^{s-1} dv\, dv_* + C\, \|f^\varepsilon\|_{L^1_2}\, \|f^\varepsilon\|_{L^1_{2s+\gamma}}, \end{aligned}$$

which gives (4.15).

Let $s \le 1$ and $\delta > 0$. We deduce from Lemma 4.11 with $s = 1+\delta/2$, Young's inequality, (4.11) and (4.12) that

$$\int_0^t \iint f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\,\frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\,|v_*|^2\left(1+|v|^2\right)^{s-1} dv\,dv_*\,d\tau$$

$$\le \int_0^t \iint f^\varepsilon f^\varepsilon_*(1-f^\varepsilon_*)\,\frac{\Psi_\varepsilon(|v-v_*|)}{|v-v_*|^2}\left(1+|v_*|^{2+\delta}\right) dv\,dv_*\,d\tau$$

$$\le C\,T\,\|f^\varepsilon\|^2_{L^\infty(0,T;L^1_2)} + \Gamma(\|f_{in}\|_{L^1_{2+\delta}}).$$

The formula (4.16) then follows directly from (4.15). □

4.3 Proof of Theorem 2.2

Consider f_{in} satisfying (2.3) and such that $f_{in} \in L^1_{2s_0}(\mathbb{R}^3)$, for some $s_0 > 1$. There exists a sequence of functions $(f_{in,k})_{k\ge1}$ in $\mathcal{C}^\infty(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)\cap W^{3,\infty}(\mathbb{R}^3)$ such that $f_{in,k} \to f_{in}$ in $L^1_{2s_0}(\mathbb{R}^3)$ and

$$C'_k\,e^{-\delta'_k\,|v|^2} \le f_{in,k} \le \frac{C_k\,e^{-\delta_k\,|v|^2}}{1+C_k\,e^{-\delta_k\,|v|^2}},$$

for some positive constants C_k, C'_k, δ_k and δ'_k.

For every $k \ge 1$, we set

$$\varepsilon_k = \frac{1}{k} \qquad \text{and} \qquad f_k = f^{\varepsilon_k},$$

where f^{ε_k} denotes the solution to (4.1) with initial datum $f_{in,k}$ given by Theorem 4.2.

Lemma 4.15 *There are a nonnegative function $f \in \mathcal{C}_w([0,T];L^2(\mathbb{R}^3))\cap L^\infty((0,T)\times\mathbb{R}^3)$ and a subsequence of $(f_k)_{k\ge1}$ (not relabeled) which converges to f in $L^2(0,T;L^1(\mathbb{R}^3))$, in $\mathcal{C}_w([0,T];L^2(\mathbb{R}^3))$ and a.e. on $(0,T)\times\mathbb{R}^3$.*

In addition, $0 \le f \le 1$ a.e. on $(0,T)\times\mathbb{R}^3$.

Here $\mathcal{C}_w([0,T];L^2(\mathbb{R}^3))$ denotes the space of weakly continuous functions in $L^2(\mathbb{R}^3)$. Since $0 \le f_k \le 1$, it follows from Lemma 4.15 and Hölder's inequality that $(f_k)_{k\ge1}$ converges to f in $L^p((0,T)\times\mathbb{R}^3)$ for any $p \in [1,\infty[$.

Proof. For $m \ge 4$ and $r > 0$, we have

$$H^1_{2s_0-2-\gamma}(\mathbb{R}^3)\cap L^1_r(\mathbb{R}^3) \subset L^1(\mathbb{R}^3) \subset \left(H^m_{2+2\gamma}(\mathbb{R}^3)\right)',$$

the embedding of $H^1_{2s_0-2-\gamma}(\mathbb{R}^3)\cap L^1_r(\mathbb{R}^3)$ in $L^1(\mathbb{R}^3)$ being compact. Since the sequence $(f_{in,k})_{k\ge1}$ converges to f_{in} in $L^1_{2s_0}(\mathbb{R}^3)$, there exists κ_0 such that $\|f_{in,k}\|_{L^1_{2s_0}} \le \kappa_0$, and

$$\Gamma(\|f_{in,k}\|_{L^1_{2s_0}}) \le \Gamma(\kappa_0), \qquad \text{for} \quad k \ge 1, \tag{4.17}$$

by Remark 4.12. We then deduce from Lemma 4.11 and Corollary 4.14 that

$$(f_k)_{k\ge1} \text{ is bounded in } L^2\Big(0,T;H^1_{2s_0-2-\gamma}(\mathbb{R}^3)\cap L^1_2(\mathbb{R}^3)\Big). \tag{4.18}$$

Next, for $\varphi \in H^m_{2+2\gamma}(\mathbb{R}^3)$, we have

$$\int \partial_t f_k\varphi\,dv = \sum_{i,j}\iint a^{\varepsilon_k}_{i,j}(v-v_*)f_kf_{k*}(1-f_{k*})\,\partial^2_{i,j}\varphi\,dv\,dv_* + \varepsilon_k\int f_k\Delta\varphi\,dv$$

$$+\sum_i\iint b^{\varepsilon_k}_i(v-v_*)f_kf_{k*}(1-f_k)\,(\partial_i\varphi-\partial_i\varphi_*)\,dv\,dv_*. \tag{4.19}$$

Hence,

$$\left|\int \partial_t f_k \varphi \, dv\right| \leq C\,\|\varphi\|_{W^{2,\infty}} \iint \frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2}\,|v_*|^2 f_k f_{k*}(1-f_{k*}) dv\,dv_* \\ + C\,\|\varphi\|_{W^{2,\infty}}\,\|f_k\|_{L^1_2}^2 + C\,\|\varphi\|_{H^2_{2+2\gamma}}\,\|f_k\|_{L^1_2}^{3/2} + \varepsilon_k\,\|\varphi\|_{W^{2,\infty}}\,\|f_{in}\|_{L^1}.$$

Since $m \geq 4$, we infer from Lemma 4.11 and the continuous embedding of $H^m(\mathbb{R}^3)$ into $W^{2,\infty}(\mathbb{R}^3)$ that

$$(\partial_t f_k)_{k\geq 1} \text{ is bounded in } L^1\left(0,T;\left(H^m_{2+2\gamma}(\mathbb{R}^3)\right)'\right). \tag{4.20}$$

By [27, Corollary 4], we conclude from (4.18) and (4.20) that $(f_k)_{k\geq 1}$ is relatively compact in the space $L^2(0,T;L^1(\mathbb{R}^3))$. Therefore, there are a function $f \in L^2(0,T;L^1(\mathbb{R}^3))$ and a subsequence of $(f_k)_{k\geq 1}$ (not relabeled) such that $(f_k)_{k\geq 1}$ converges towards f in $L^2\big(0,T;L^1(\mathbb{R}^3)\big)$ and a.e. on $(0,T)\times\mathbb{R}^3$.

Moreover, we deduce from (4.19) that, for $\varphi \in \mathcal{C}_0^2(\mathbb{R}^3)$ with compact support included in B_R for some $R>0$, we have

$$\left|\int f_k(t)\varphi\,dv - \int f_k(\sigma)\varphi\,dv\right| \\ \leq C\,\|\varphi\|_{W^{2,\infty}} \int_\sigma^t \iint_{|v|\leq R} \frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2}\,|v_*|^2\, f_k f_{k*}(1-f_{k*}) dv\,dv_*\,d\tau \\ + C\,\|\varphi\|_{W^{2,\infty}}\,|t-\sigma| \left[\|f_k\|^2_{L^\infty(0,T;L^1_2)} + \|f_k\|^{3/2}_{L^\infty(0,T;L^1_2)} + \|f_{in}\|_{L^1}\right]. \tag{4.21}$$

From Lemma 4.11, we deduce that, for $R'>0$,

$$\int_0^T \iint_{|v|\leq R',\,|v_*|\leq R'} \frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2}\,|v_*|^{2s_0}\, f_k f_{k*}(1-f_{k*})\,dv\,dv_*\,d\tau \leq \Gamma(\|f_{in,k}\|_{L^1_{2s_0}}) \leq \Gamma(\kappa_0).$$

We may then pass to the limit as $k\to+\infty$ thanks to the a.e. convergence of $(f_k)_{k\geq 1}$ and $(\Psi_{\varepsilon_k})_{k\geq 1}$ and then as $R'\to+\infty$ by the Fatou lemma to obtain

$$\int_0^T \iint |v-v_*|^\gamma\,|v_*|^{2s_0}\, f f_*(1-f_*)\,dv\,dv_*\,d\tau \leq \Gamma(\kappa_0). \tag{4.22}$$

Next, it is easy to check, by means of the a.e. convergence and (4.22), that

$$\left(\iint_{|v|\leq R} \frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2} f_k f_{k*}\,(1-f_{k*})\;|v_*|^2\,dv\,dv_*\right)_{k\geq 1}$$

converges towards

$$\iint_{|v|\leq R} |v-v_*|^\gamma\,|v_*|^2\, f f_*(1-f_*)\,dv\,dv_*$$

in $L^1(0,T)$. Therefore, the Vitali theorem implies that

$$\lim_{|t-\sigma|\to 0}\,\sup_{k\geq 1} \int_\sigma^t \iint_{|v|\leq R} f_k f_{k*}\,(1-f_{k*})\,\frac{\Psi_{\varepsilon_k}(|v-v_*|)}{|v-v_*|^2}\,|v_*|^2\,dv\,dv_*\,d\sigma = 0.$$

We then deduce from (4.21) that the sequence $(\int f_k\varphi\,dv)_{k\geq 1}$ is equicontinuous and bounded in $\mathcal{C}([0,T])$. The Arzela-Ascoli theorem ensures that it is relatively compact in $\mathcal{C}([0,T])$. From the convergence of $(f_k)_{k\geq 1}$ towards f in $L^1\big((0,T)\times\mathbb{R}^3\big)$, we deduce that $\int f\,\varphi\,dv$ is the unique cluster point of $(\int f_k\varphi\,dv)_{k\geq 1}$. Therefore, $(\int f_k\varphi\,dv)_{k\geq 1}$ converges to $\int f\varphi\,dv$ in $\mathcal{C}([0,T])$. Since the sequence $(f_k)_{k\geq 1}$ and its limit f are bounded in $L^\infty(0,T;L^2(\mathbb{R}^3))$, it follows that $(f_k)_{k\geq 1}$ converges towards f in $\mathcal{C}_w([0,T];L^2(\mathbb{R}^3))$. □

Lemma 4.16 *The limit f of the sequence $(f_k)_{k\geq 1}$ is a solution to the Landau-Fermi-Dirac equation (2.1), (2.2) which satisfies (2.4) and (2.5).*

Proof. *Step 1: Conservation of mass and energy.*

Let $t\in[0,T]$. By Lemma 4.11 and (4.17), we have

$$\int_{|v|\leq R} f_k(t,v)\,|v|^{2s_0}\,dv \leq \int f_k(t,v)\,|v|^{2s_0}\,dv \leq \Gamma(\kappa_0),$$

for each $k\geq 1$. Thanks to Lemma 4.15 and the Fatou lemma, we may let $k\to+\infty$ and then $R\to+\infty$ and obtain

$$\int f(t,v)|v|^{2s_0}dv \leq \Gamma(\kappa_0). \tag{4.23}$$

Combining Lemma 4.11, Lemma 4.15 and (4.23), we see that $(M_{2r}(f_k))_{k\geq 1}$ converges strongly towards $M_{2r}(f)$ in $\mathcal{C}([0,T])$ for $r\in[0,s_0)$. Since $(f_{in,k})_{k\geq 1}$ converges to f_{in} in $L^1_{2s_0}(\mathbb{R}^3)$ and $s_0>1$, we deduce

$$\lim_{k\to+\infty}\int f_k(t,v)\,dv = \lim_{k\to+\infty}\int f_{in,k}(v)\,dv = \int f_{in}(v)\,dv,$$

$$\lim_{k\to+\infty}\int |v|^2\,f_k(t,v)\,dv = \lim_{k\to+\infty}\big(M_2(f_{in,k}) + 6\,\varepsilon_k\,M_0(f_{in,k})t\big) = M_2(f_{in}).$$

We thus conclude that f conserves mass and energy.

Step 2: Passage to the limit in the weak formulation (4.9).

For all $k\geq 1$, $\varphi\in\mathcal{C}^2_b(\mathbb{R}^3)$ and $t\in[0,T]$, the functions f_k satisfy,

$$\begin{aligned}\int f_k(t,v)\,\varphi(v)\,dv &- \int f_{in,k}(v)\,\varphi(v)\,dv \\ &= \sum_{i,j}\int_0^t d\sigma \iint a^{\varepsilon_k}_{i,j}(v-v_*)f_k f_{k*}\,(1-f_{k*})\,\partial^2_{i,j}\varphi\,dv\,dv_* + \varepsilon_k\int_0^t d\sigma\int f_k\Delta\varphi\,dv \\ &\quad + \sum_i\int_0^t d\sigma\iint b^{\varepsilon_k}_i(v-v_*)\,f_k f_{k*}\,(2-f_k-f_{k*})\,\partial_i\varphi\,dv\,dv_*. \end{aligned}\tag{4.24}$$

Our aim is here to pass to the limit as $k\to+\infty$ in formula (4.24). By Lemma 4.15, it is obvious for the left-hand side and the second integral in the right-hand side. We thus have to consider the two remaining integrals. As $(\Psi_{\varepsilon_k})_{k\geq 1}$ converges pointwisely towards Ψ, the functions $a^{\varepsilon_k}_{i,j}$ and $b^{\varepsilon_k}_i$ defined at the beginning of Section 4.1 converge towards $a_{i,j}$ and b_i respectively. Consider $\varphi\in\mathcal{C}^2(\mathbb{R}^3)$ with compact support included in B_R for some

$R > 0$. Let $R' > 0$. We first turn our attention to the integral involving the matrix a^{ε_k}.

$$
\begin{aligned}
&\Big| \sum_{i,j} \int_0^t d\sigma \iint \Big[a_{i,j}^{\varepsilon_k}(v-v_*) f_k f_{k*} (1-f_{k*}) - a_{i,j}(v-v_*) f f_* (1-f_*) \Big] \partial^2_{i,j}\varphi \, dv \, dv_* \Big| \\
&\leq \Big| \sum_{i,j} \int_0^t d\sigma \iint_{B_R \times B_{R'}} \Big[a_{i,j}^{\varepsilon_k}(v-v_*) f_k f_{k*} (1-f_{k*}) - a_{i,j}(v-v_*) f f_* (1-f_*) \Big] \partial^2_{i,j}\varphi \, dv \, dv_* \Big| \\
&\quad + C \, \|\varphi\|_{W^{2,\infty}} \int_0^t d\sigma \iint_{\{|v|\leq R, \, |v_*| \geq R'\}} \Psi_{\varepsilon_k}(|v-v_*|) f_k f_{k*} (1-f_{k*}) \, dv \, dv_* \\
&\quad + C \, \|\varphi\|_{W^{2,\infty}} \int_0^t d\sigma \iint_{\{|v|\leq R, \, |v_*| \geq R'\}} |v-v_*|^{\gamma+2} f f_* (1-f_*) \, dv \, dv_*. \qquad (4.25)
\end{aligned}
$$

The a.e. convergence of a^{ε_k} and f_k, the bound on f_k, the properties of Ψ_{ε_k} and the Lebesgue dominated convergence theorem imply that the first term of the right-hand side of (4.25) converges to zero. For the two others, it follows from (4.14) and (4.22) that

$$
\begin{aligned}
&\int_0^t d\sigma \iint_{\{|v|\leq R, \, |v_*| \geq R'\}} |v-v_*|^{\gamma+2} f f_* (1-f_*) \, dv \, dv_* \\
&\qquad \leq 2 \left[R^2 R'^{-2s_0} + R'^{2-2s_0} \right] \int_0^t d\sigma \iint |v-v_*|^{\gamma} |v_*|^{2s_0} f f_* (1-f_*) \, dv \, dv_* \\
&\qquad \leq 2 \, \Gamma(\kappa_0) \left[R^2 R'^{-2s_0} + R'^{2-2s_0} \right],
\end{aligned}
$$

and

$$
\int_0^t d\sigma \iint_{\{|v|\leq R, \, |v_*| \geq R'\}} \Psi_{\varepsilon_k}(|v-v_*|) f_k f_{k*} (1-f_{k*}) \, dv \, dv_* \leq 2 \, \Gamma(\kappa_0) \big[R^2 R'^{-2s_0} + R'^{2-2s_0} \big].
$$

We then substitute these estimates in (4.25) and let first $k \to +\infty$ and then $R' \to +\infty$ to obtain that the left-hand side converges to zero as $k \to +\infty$.

We proceed analogously for the integral of (4.24) which involves the function b^{ε_k}.

$$
\begin{aligned}
&\Big| \sum_{i} \int_0^t d\sigma \iint \Big[b_i^{\varepsilon_k}(v-v_*) \, f_k f_{k*} (2-f_k-f_{k*}) - b_i(v-v_*) \, f f_* (2-f-f_*) \Big] \partial_i\varphi \, dv \, dv_* \Big| \\
&\qquad \leq \Big| \sum_{i} \int_0^t d\sigma \iint_{B_R \times B_{R'}} \Big[b_i^{\varepsilon_k} \, f_k f_{k*} (2-f_k-f_{k*}) - b_i \, f f_* (2-f-f_*) \Big] \partial_i\varphi \, dv \, dv_* \Big| \\
&\qquad\quad + C \, \|\varphi\|_{W^{2,\infty}} \int_0^t d\sigma \iint_{\{|v|\leq R, \, |v_*| \geq R'\}} \frac{\Psi_{\varepsilon_k}(v-v_*)}{|v-v_*|} f_k f_{k*} \, dv \, dv_* \\
&\qquad\quad + C \, \|\varphi\|_{W^{2,\infty}} \int_0^t d\sigma \iint_{\{|v|\leq R, \, |v_*| \geq R'\}} |v-v_*|^{1+\gamma} \, f f_* \, dv \, dv_*. \qquad (4.26)
\end{aligned}
$$

For the first term of the right-hand side, we use again the a.e. convergence of a^{ε_k} and f_k, the bound on f_k, the properties of Ψ_{ε_k} and the Lebesgue dominated convergence theorem, whereas for the two others, we have

$$
\begin{aligned}
&\int_0^t d\sigma \iint_{\{|v|\leq R, \, |v_*| \geq R'\}} |v-v_*|^{1+\gamma} \, f f_* \, dv \, dv_* \\
&\qquad \leq C \int_0^t d\sigma \iint_{\{|v|\leq R, \, |v_*| \geq R'\}} \left(1+|v|^2\right)^{(1+\gamma)/2} \left(1+|v_*|^2\right)^{(1+\gamma)/2} f f_* \, dv \, dv_* \\
&\qquad \leq C \, T \, (1+R'^2)^{(1+\gamma-2s_0)/2} \, \|f\|_{L^1_2} \, \|f\|_{L^1_{2s_0}},
\end{aligned}
$$

and

$$\int_0^t d\sigma \iint_{\{|v|\le R, |v_*|\ge R'\}} \frac{\Psi_{\varepsilon_k}(v-v_*)}{|v-v_*|}\, f_k f_{k*}\, dv\, dv_* \le C\,T\,(1+R'^2)^{(1+\gamma-2s_0)/2}\, \|f_k\|^2_{L^1_{2s_0}}$$
$$\le C\,T\,(1+R'^2)^{(1+\gamma-2s_0)/2}\, \Gamma(\kappa_0)^2,$$

by Lemma 4.11. Inserting the estimates in (4.26) and letting first $k \to +\infty$ and then $R' \to +\infty$, we obtain that the left-hand side converges to zero as $k \to +\infty$.

Therefore, f is a weak solution to the Landau-Fermi-Dirac equation (2.1), (2.2) which preserves mass and energy. □

Moreover, we deduce from (4.22) and (4.23) that f satisfies

$$f(1-f) \in L^1_{loc}\big(\mathbb{R}_+; L^1_{2s_0+\gamma}(\mathbb{R}^3)\big) \quad \text{and} \quad f \in L^\infty_{loc}\big(\mathbb{R}_+; L^1_{2s_0}(\mathbb{R}^3)\big). \tag{4.27}$$

Distinguishing the cases $s_0 < 1+\gamma/2$ and $s_0 \ge 1+\gamma/2$, we infer from (4.15) and (4.16) the existence of a constant $C(T,\kappa_0)$ such that, for all $R>0$, $k \ge R/2$,

$$\left(\frac{1}{2}\right)^{2+\gamma} \int_0^T \int_{|v|\le R} |\nabla f_k|^2 \left(1+|v|^2\right)^{s_0}\, dv\, d\tau \le (1+\varepsilon_k)\, C(T,\kappa_0).$$

Letting first $k \to +\infty$ thanks to a weak compactness argument and then $R \to +\infty$ by the Fatou lemma, we conclude that

$$\nabla f \in L^2_{loc}\big(\mathbb{R}_+; L^2_{2s_0}(\mathbb{R}^3)\big). \tag{4.28}$$

Therefore, the proof of the first statement of Theorem 2.2 is now complete.

We now verify that the entropy of f is a non-decreasing function when $f_{in} \in L^1_{2+\gamma}(\mathbb{R}^3)$, which corresponds to the second statement of Theorem 2.2. For that purpose, we first need a smoothness result.

Lemma 4.17 *Let $f_{in} \in L^1_{2+\gamma}(\mathbb{R}^3)$ satisfying (2.3). The weak solution f to (2.1), (2.2) given by Lemma 4.16 belongs to $\mathcal{C}([0,T]; L^2(\mathbb{R}^3))$.*

Proof. Let us first show that

$$\partial_t f \in L^2\Big(0,T; \big(H^1_{2+\gamma}(\mathbb{R}^3)\big)'\Big). \tag{4.29}$$

Indeed, the function f satisfies, in the sense of distributions,

$$\partial_t f = \nabla \cdot \Big[\overline{A}\nabla f - \overline{b} f(1-f)\Big].$$

Moreover, since the initial datum belongs to $L^1_{2+\gamma}(\mathbb{R}^3)$, we have

$$f \in L^\infty\big(0,T; L^1_{2+\gamma}(\mathbb{R}^3)\big) \cap L^2\big(0,T; H^1_{2+\gamma}(\mathbb{R}^3)\big),$$

by (4.27) and (4.28). Consequently,

$$\|\nabla \cdot \big[\overline{A}\nabla f\big]\|_{(H^1_{2+\gamma})'} \le C\|f\|_{L^1_{2+\gamma}}\|f\|_{H^1_{2+\gamma}},$$
$$\|\nabla \cdot \big[\overline{b} f(1-f)\big]\|_{(H^1_{2+\gamma})'} \le C\|f\|^2_{L^1_2},$$

whence (4.29). Since

$$H^1_{2+\gamma}(\mathbb{R}^3) \subset L^2_{2+\gamma}(\mathbb{R}^3) \subset (H^1_{2+\gamma}(\mathbb{R}^3))',$$

with continuous and dense embeddings, and

$$f \in L^2(0,T;H^1_{2+\gamma}(\mathbb{R}^3)) \quad \text{ and } \quad \partial_t f \in L^2(0,T;(H^1_{2+\gamma}(\mathbb{R}^3))'),$$

we have $f \in \mathcal{C}([0,T];L^2_{2+\gamma}(\mathbb{R}^3))$ by [21, Proposition 1.2.1 and Theorem 1.3.1] (see also [14, Theorem 5.9.3]). Lemma 4.17 then follows since $L^2_{2+\gamma}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$. □

Lemma 4.18 *Let $f_{in} \in L^1_{2+\gamma}(\mathbb{R}^3)$ satisfying (2.3). Let f denote the weak solution to (2.1), (2.2) given by Lemma 4.16. The entropy $S(f)$ is a continuous and non-decreasing function such that, for $t \ge 0$,*

$$S(f_{in}) \le S(f)(t) \le E_{in} + \int e^{-|v|^2}\, dv. \tag{4.30}$$

Proof. We first show the continuity of $S(f)$. Let $t \ge 0$ and $(t_n)_{n\ge1}$ be a sequence converging to t. Lemma 4.17 implies that $(f(t_n))_{n\ge1}$ converges towards $f(t)$ in $L^2(\mathbb{R}^3)$. One can extract a subsequence $f(t_{\varphi(n)})_{n\ge1}$ which converges a.e. in $\mathbb{R}^3$ towards $f(t)$.

From the inequality

$$s(r) \le r|v|^2 + e^{-|v|^2} \qquad \text{for } 0 \le r \le 1, \tag{4.31}$$

where $s(r) = r|\ln r| + (1-r)|\ln(1-r)|$, we deduce that

$$\begin{aligned} &\Big|S(f)(t_{\varphi(n)}) - S(f)(t)\Big| \\ &\le \left|\int_{|v|\le R} \Big(s(f)(t_{\varphi(n)}) - s(f)(t)\Big)\, dv\right| + \int_{|v|\ge R} \Big(f(t_{\varphi(n)}) + f(t)\Big)|v|^2 dv + 2\int_{|v|\ge R} e^{-|v|^2} dv \\ &\le \left|\int_{|v|\le R} \Big(s(f)(t_{\varphi(n)}) - s(f)(t)\Big)\, dv\right| + 2R^{-\gamma}\,\Gamma(\kappa_0) + 2\int_{|v|\ge R} e^{-|v|^2} dv, \end{aligned}$$

hence the convergence of $(S(f)(t_{\varphi(n)}))_{n\ge1}$ towards $S(f)(t)$. Since $(S(f)(t_n))_{n\ge1}$ is bounded by (2.5) and (4.31) and has a unique cluster point $S(f)(t)$, we conclude that $(S(f)(t_n))_{n\ge1}$ converges to $S(f)(t)$.

Let us now prove the monotonicity of $S(f)$. Consider $h \ge 0$ and $0 \le \sigma \le t$. We deduce from (4.13) that

$$hS(f_{in,k}) \le \int_\sigma^{\sigma+h} S(f_k)(\tau)\, d\tau \le \int_t^{t+h} S(f_k)(\tau)\, d\tau. \tag{4.32}$$

As previously, (4.31) imply that

$$\int_0^T \Big|S(f_k) - S(f)\Big|\, dt \le \int_0^T \left|\int_{B_R} \Big(s(f_k) - s(f)\Big)\, dv\right| dt + 2TR^{-\gamma}\,\Gamma(\kappa_0) + 2T\int_{|v|\ge R} e^{-|v|^2} dv,$$

and thus that $(S(f_k))_{k\ge1}$ converges to $S(f)$ in $L^1(0,T)$. Similarly, $(S(f_{in,k}))_{k\ge1}$ converges to $S(f_{in})$. We may then pass to the limit as $k \to +\infty$ in (4.32) to obtain

$$S(f_{in}) \le \frac{1}{h}\int_\sigma^{\sigma+h} S(f)(\tau)\, d\tau \le \frac{1}{h}\int_t^{t+h} S(f)(\tau)\, d\tau.$$

Letting $h \to 0$ thanks to the continuity of $S(f)$ completes the proof of the monotonicity of $S(f)$ and the first inequality in (4.30). Finally, the second inequality in (4.30) follows from (4.31). □

5 Uniqueness

In this section, we are concerned with the uniqueness issue. As previously mentioned, we first need an embedding lemma for weighted Sobolev spaces because of the non-quadratic nature of the LFD collision operator.

Lemma 5.1 *For all $r \geq 0$, $\varepsilon > 0$, there exists a constant $C > 0$ such that, for every function $h \in H^1_{2r}(\mathbb{R}^3)$, we have*

$$\|h\|_{L^4_{2r}} \leq C\varepsilon^{-3/4}\|h\|_{L^2_{2r}} + C\varepsilon^{1/4}\|\nabla h\|_{L^2_{2r}}.$$

The proof of Lemma 5.1 is an easy extension of [26, Lemma 3.6.7] where the above inequality is established for $r = 0$.

Theorem 5.2 *Let $f_{in} \in L^1_{2s}(\mathbb{R}^3)$ with $2s > 4\gamma + 11$, satisfying (2.3). Then there is a unique weak solution f to (2.1), (2.2) (in the sense of Definition 2.1) such that*

$$f \in L^\infty_{loc}(\mathbb{R}_+; L^2_{2s}(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}_+; H^1_{2s}(\mathbb{R}^3)).$$

Remark 5.3 *Since $0 \leq f_{in} \leq 1$, f_{in} belongs to $L^2_{2s}(\mathbb{R}^3)$ as soon as it belongs to $L^1_{2s}(\mathbb{R}^3)$. Thus we do not need any extra assumption in a weighted L^2-space as in [10].*

Proof. We only give formal computations in order to highlight the difference with the proof for the classical Landau equation performed in [10, Theorem 7]. Let f_1 and f_2 be two solutions to (2.1), (2.2) satisfying the requirements of Theorem 5.2. We set $u = f_1 - f_2$ and $w = f_1 + f_2$. The function u satisfies, in the sense of distributions,

$$\partial_t u = \frac{1}{2}\nabla \cdot \Big\{\overline{A}^{u(1-w)}\,\nabla w + \big(\overline{A}^{f_1} + \overline{A}^{f_2}\big)\,\nabla u - \overline{b}^u\,\big[f_1(1-f_1) + f_2(1-f_2)\big] - \overline{b}^w\,u\,(1-w)\Big\}.$$

Then, for every $q > 0$,

$$\begin{aligned}
\frac{d}{dt}\int |u|^2(1+|v|^2)^q\,dv \;&=\; -\int \overline{A}^{u(1-w)}\nabla w\,\nabla\big[u(1+|v|^2)^q\big]dv && (5.1)\\
&\quad -\int \Big[\overline{A}^{f_1} + \overline{A}^{f_2}\Big]\nabla u\,\nabla\big[u(1+|v|^2)^q\big]dv && (5.2)\\
&\quad +\int \overline{b}^u\big[f_1(1-f_1) + f_2(1-f_2)\big]\,\nabla\big[u(1+|v|^2)^q\big]dv && (5.3)\\
&\quad +\int \overline{b}^w u\,(1-w)\,\nabla\big[u(1+|v|^2)^q\big]dv. && (5.4)
\end{aligned}$$

We first consider (5.2) and (5.4),

$$\begin{aligned}
(5.2)+(5.4) \;=\; &-q\int \Big[\overline{A}^{f_1} + \overline{A}^{f_2}\Big]\nabla\big(u^2\big)\big(1+|v|^2\big)^{q-1}v\,dv\\
&-\int \Big[\overline{A}^{f_1} + \overline{A}^{f_2}\Big]\nabla u\nabla u\,\big(1+|v|^2\big)^q\,dv + \frac{1}{2}\int \overline{b}^w\cdot\nabla\big(u^2\big)\big(1+|v|^2\big)^q\,dv\\
&-\int u\,w\,\overline{b}^w\cdot\nabla u\,(1+|v|^2)^q dv + 2q\int u^2\,(1-w)\,\overline{b}^w\cdot v\,\big(1+|v|^2\big)^{q-1}\,dv.
\end{aligned}$$

With the ellipticity of the diffusion matrix and an integration by parts in the integrals involving the term $\nabla\big(u^2\big)$, we find

$$
\begin{aligned}
(5.2)+(5.4) \;\leq\; & -2K\int |\nabla u|^2\,\big(1+|v|^2\big)^{q+\gamma/2}\,dv + \,q\int u^2\,\Big[\overline{B}^{f_1}+\overline{B}^{f_2}\Big]\cdot v\,\big(1+|v|^2\big)^{q-1}\,dv \\
& + q\int u^2\Big[\overline{A}^{f_1}+\overline{A}^{f_2}\Big] : \nabla\big[\big(1+|v|^2\big)^{q-1}v\big]dv - \frac{1}{2}\int u^2\,\overline{c}^w\big(1+|v|^2\big)^q\,dv \\
& - q\int u^2\,\overline{b}^w\cdot v\,\big(1+|v|^2\big)^{q-1}\,dv - \int u\,w\,\overline{b}^w\cdot\nabla u\,(1+|v|^2)^q\,dv \\
& + 2q\int u^2\,\overline{b}^w\cdot v\,\big(1+|v|^2\big)^{q-1}\,dv - \,2q\int u^2\,w\,\overline{b}^w\cdot v\,\big(1+|v|^2\big)^{q-1}\,dv.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(5.2)+(5.4) \;\leq\; & -2K\int |\nabla u|^2\,\big(1+|v|^2\big)^{q+\gamma/2}\,dv + \int u^2\,E\,dv \\
& + \left|\int u\,w\,\overline{b}^w\cdot\nabla u\,(1+|v|^2)^q dv\right| + 2q\left|\int u^2\,w\,\overline{b}^w\cdot v\,\big(1+|v|^2\big)^{q-1}\,dv\right|,
\end{aligned}
$$

where

$$
E = q\Big[\overline{B}^{f_1}+\overline{B}^{f_2}+\overline{b}^w\Big]\cdot v\,\big(1+|v|^2\big)^{q-1} + q\Big[\overline{A}^{f_1}+\overline{A}^{f_2}\Big] : \nabla\big[\big(1+|v|^2\big)^{q-1}v\big] - \frac{1}{2}\,\overline{c}^w\big(1+|v|^2\big)^q.
$$

Now,

$$
\begin{aligned}
E \;=\; & q\int \big[f_{1*}(1-f_{1*})+f_{2*}(1-f_{2*})\big]\,|v-v_*|^\gamma\big(1+|v|^2\big)^{q-2} \\
& \times\Big\{-2|v|^2(v\cdot v_*)+2q|v|^2|v_*|^2-2(q-1)(v\cdot v_*)^2-2\,v\cdot v_*+2\,|v_*|^2\Big\}\,dv_* \\
& + \int w_*|v-v_*|^\gamma\big(1+|v|^2\big)^{q-1}\Big\{(\gamma+3-2q)|v|^2+2\,q\,(v\cdot v_*)+\gamma+3\Big\}\,dv_*,
\end{aligned}
$$

and choosing $2q>\gamma+3$, we deduce that $E\leq C\big(1+|v|^2\big)^q$ since $f_i\in L^\infty_{loc}(\mathbb{R}_+;L^1_{\gamma+2}(\mathbb{R}^3))$ for $i=1,2$. Consequently,

$$
\begin{aligned}
(5.2)+(5.4) \;\leq\; & -2K\int |\nabla u|^2\,\big(1+|v|^2\big)^{q+\gamma/2}\,dv \,+\, C\int u^2\,\big(1+|v|^2\big)^q\,dv \\
& + \left|\int u\,w\,\overline{b}^w\cdot\nabla u\,(1+|v|^2)^q dv\right| +\, 2q\left|\int u^2\,w\,\overline{b}^w\cdot v\,\big(1+|v|^2\big)^{q-1}dv\right| \qquad (5.5)
\end{aligned}
$$

From Hölder's inequality and Lemma 5.1, we deduce that for every $\varepsilon>0$,

$$
\begin{aligned}
\left|\int u\,w\,\overline{b}^w\cdot\nabla u\,(1+|v|^2)^q\,dv\right| \;\leq\; & C\|w\|^{1/2}_{L^2_{2q+2\gamma+4}}\,\|\nabla u\|_{L^2_{2q+\gamma}}\,\|u\|_{L^4_{2q}} \\
\leq\; & C_\varepsilon\|w\|^4_{L^2_{2q+2\gamma+4}}\|u\|^2_{L^2_{2q}} + \varepsilon\,\|\nabla u\|^2_{L^2_{2q+\gamma}}, \qquad (5.6)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int u^2\,w\,\overline{b}^w\cdot v\,\big(1+|v|^2\big)^{q-1}\,dv\right| \;\leq\; & C\|w\|_{L^2_{2q+2\gamma}}\,\|u\|^2_{L^4_{2q}} \\
\leq\; & C_\varepsilon\|w\|^4_{L^2_{2q+2\gamma}}\,\|u\|^2_{L^2_{2q}} + \varepsilon\,\|\nabla u\|^2_{L^2_{2q+\gamma}}. \qquad (5.7)
\end{aligned}
$$

Finally, substituting (5.6) and (5.7) in (5.5), we find

$$(5.2) + (5.4) \leq -2\,(K-\varepsilon)\,\|\nabla u\|^2_{L^2_{2q+\gamma}} \; + C_\varepsilon \|u\|^2_{L^2_{2q}} \left(1 + \|w\|^4_{L^2_{2q+2\gamma+4}}\right). \tag{5.8}$$

It remains now to consider (5.1) and (5.3). In the sequel, we use the notation Π for $\Pi(v-v_*)$. We have

$$(5.1) + (5.3) = -\iint \Pi |v-v_*|^{\gamma+2} u_* (1-w_*) \nabla w \nabla u\,(1+|v|^2)^q\,dv\,dv_* \tag{5.9}$$

$$-2\,q \iint \Pi |v-v_*|^{\gamma+2} u_* (1-w_*) \nabla w\; u\; v\,(1+|v|^2)^{q-1}\,dv\,dv_* \tag{5.10}$$

$$-2 \iint (v-v_*)\cdot\nabla u\,|v-v_*|^\gamma u_* \big[f_1(1-f_1)+f_2(1-f_2)\big]\,(1+|v|^2)^q\,dv\,dv_* \tag{5.11}$$

$$-4\,q \iint (v-v_*)\cdot v\,|v-v_*|^\gamma u_* \big[f_1(1-f_1)+f_2(1-f_2)\big] u\,(1+|v|^2)^{q-1}\,dv\,dv_* \tag{5.12}$$

Using successively the Cauchy-Schwarz inequality, $|v-v_*|^{2r} \leq C\big(1+|v|^2\big)^r\big(1+|v_*|^2\big)^r$ and the Fubini theorem, we find

$$\begin{aligned}
(5.9) \quad &\leq \quad C\left\{\iint |v-v_*|^{\gamma+2}\,|u_*|\,|\nabla u|^2\,\big(1+|v|^2\big)^{q-1}\,dv\,dv_*\right\}^{1/2} \\
&\qquad \times \left\{\iint |v-v_*|^{\gamma+2}\,|u_*|\,|\nabla w|^2\,\big(1+|v|^2\big)^{q+1}\,dv\,dv_*\right\}^{1/2} \\
&\leq \quad C\,\|u\|_{L^1_{\gamma+2}}\,\|\nabla u\|_{L^2_{2q+\gamma}}\,\|\nabla w\|_{L^2_{2q+\gamma+4}},
\end{aligned}$$

$$\begin{aligned}
(5.10) \quad &\leq \quad C\left\{\iint |v-v_*|^{\gamma+2}\,|v|^2\,|u_*|\,|\nabla w|^2\,\big(1+|v|^2\big)^{q-1+\gamma/2}\,dv\,dv_*\right\}^{1/2} \\
&\qquad \times \left\{\iint |v-v_*|^{\gamma+2}\,|u_*|\,|u|^2\,\big(1+|v|^2\big)^{q-1-\gamma/2}\,dv\,dv_*\right\}^{1/2} \\
&\leq \quad C\,\|u\|_{L^1_{\gamma+2}}\,\|u\|_{L^2_{2q}}\,\|\nabla w\|_{L^2_{2q+2\gamma+2}},
\end{aligned}$$

$$\begin{aligned}
(5.11) \quad &\leq \quad C\left\{\iint |v-v_*|^{\gamma+2}\,|u_*|\,|w|^2\,\big(1+|v|^2\big)^{q}\,dv\,dv_*\right\}^{1/2} \\
&\qquad \times \left\{\iint |v-v_*|^{\gamma}\,|u_*|\,|\nabla u|^2\,\big(1+|v|^2\big)^{q}\,dv\,dv_*\right\}^{1/2} \\
&\leq \quad C\,\|u\|_{L^1_{\gamma+2}}\,\|\nabla u\|_{L^2_{2q+\gamma}}\,\|w\|_{L^2_{2q+\gamma+2}},
\end{aligned}$$

$$\begin{aligned}
(5.12) \quad &\leq \quad C\left\{\iint |v-v_*|^{\gamma+2}\,|u_*|\,|w|^2\,\big(1+|v|^2\big)^{q-1+\gamma/2}\,dv\,dv_*\right\}^{1/2} \\
&\qquad \times \left\{\iint |v-v_*|^{\gamma}\,|v|^2\,|u_*|\,|u|^2\,\big(1+|v|^2\big)^{q-1-\gamma/2}\,dv\,dv_*\right\}^{1/2} \\
&\leq \quad C\,\|u\|_{L^1_{\gamma+2}}\,\|u\|_{L^2_{2q}}\,\|w\|_{L^2_{2q+2\gamma}}.
\end{aligned}$$

Since $\gamma \leq 1$, we thus obtain

$$\begin{aligned}
(5.1) + (5.3) \quad &\leq \quad C\,B(t)\,\|u\|_{L^1_{\gamma+2}}\,\|\nabla u\|_{L^2_{2q+\gamma}} + C\,B(t)\,\|u\|_{L^1_{\gamma+2}}\,\|u\|_{L^2_{2q}} \\
&\qquad + C\,A\,\|u\|_{L^1_{\gamma+2}}\,\|\nabla u\|_{L^2_{2q+\gamma}} + C\,A\,\|u\|_{L^1_{\gamma+2}}\,\|u\|_{L^2_{2q}}, \\
&\leq \quad C\,\big(A+B(t)\big)\,\|u\|_{L^1_{\gamma+2}}\left(\|\nabla u\|_{L^2_{2q+\gamma}} + \|u\|_{L^2_{2q}}\right),
\end{aligned}$$

where $A = \sup_{t\in[0,T]} \|w(t)\|_{L^2_{2q+2\gamma+4}}$ and $B(t) = \|\nabla w(t)\|_{L^2_{2q+\gamma+4}}$.

Now, for $\delta > 0$, we have $\|u\|_{L^1_{\gamma+2}} \leq C_\delta \|u\|_{L^2_{2\gamma+7+\delta}}$, and thus, for $2q > 2\gamma + 7$,

$$(5.1) + (5.3) \leq \varepsilon \, \|\nabla u\|^2_{L^2_{2q+\gamma}} + C_\varepsilon \, (1 + A^2 + B^2(t)) \|u\|^2_{L^2_{2q}}. \tag{5.13}$$

From (5.8) and (5.13), we infer that

$$\|u\|^2_{L^2_{2q}}(t) \leq C \int_0^t \big(1 + A^2 + A^4 + B^2(\tau)\big) \; \|u\|^2_{L^2_{2q}}(\tau) \, d\tau.$$

Since A is finite and B belongs to $L^2_{loc}(\mathbb{R}_+)$, we may use the Gronwall lemma and conclude that $u = 0 = f_1 - f_2$. □

A Well-posedness of (4.3)

We give here further details for the proof of the well-posedness statement of Theorem 4.3. In order to apply [17, Theorem 5.8.1], we introduce the quasilinear problem

$$\left\{ \begin{array}{l} \partial_t f = \nabla \cdot \Big((\overline{A}^{g,\varepsilon} + \varepsilon I_3) \nabla f \Big) - (1 - 2f) \, \overline{b}^{g,\varepsilon} \cdot \nabla f - \overline{c}^{g,\varepsilon} f \, \theta(f) \\ f(0, .) = f_{in}. \end{array} \right. \tag{A.1}$$

where the function θ is defined on $\mathbb{R}$ by

$$\theta(f) = \left\{ \begin{array}{ccc} 1 & \text{if} & f \leq 0 \\ 1 - f & \text{if} & 0 \leq f \leq 1 \\ 0 & \text{if} & f \geq 1 \end{array} \right.$$

Let $\delta \in (0, 1)$. Then, $f_{in} \in \mathcal{H}^{2+\delta}(\mathbb{R}^3)$. Owing to Lemma 4.4 and the uniform ellipticity (4.5), the functions α_i and α defined by

$$\alpha_i(t, v, \xi) = \sum_j (\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon \delta_{i,j}) \, \xi_j \quad \text{ and } \quad \alpha(t, v, f, \xi) = (1 - 2f) \sum_k \overline{b}^{g,\varepsilon}_k \xi_k + \overline{c}^{g,\varepsilon} f \, \theta(f)$$

satisfy the assumptions of [17, Theorem 5.8.1], which implies the existence of a solution f^ε to (A.1) belonging to the Hölder space $\mathcal{H}^{2+\delta, (2+\delta)/2}([0, T] \times \mathbb{R}^3)$. Moreover, there exists a constant Λ depending only on f_{in}, δ, T, ε and C_L such that

$$\|f^\varepsilon\|_{\mathcal{H}^{2+\delta,(2+\delta)/2}} \leq \Lambda.$$

It remains to prove that $0 \leq f^\varepsilon(t, v) \leq 1$. To this aim, we consider the linear operator $\mathcal{L}_1$ defined by

$$\mathcal{L}_1 u = \partial_t u - \sum_{i,j} \Big(\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon \, \delta_{i,j} \Big) \, \partial^2_{i,j} u - \sum_i \Big[\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i (1 - 2f^\varepsilon) \Big] \partial_i u + \overline{c}^{g,\varepsilon} \, \theta(f^\varepsilon) \, u.$$

Let $R > 0$. As soon as

$$C \geq 6K_\varepsilon \|f_{in}\|_{L^1} + 6\varepsilon + 12 \, (1 + \Lambda)^2 K^2_\varepsilon \|f_{in}\|^2_{L^1} \quad \text{ and } \quad \lambda \geq 1 + K_\varepsilon \|f_{in}\|_{L^1},$$

we deduce from the comparison principle [17, Theorem 1.2.1] that

$$f^\varepsilon(t, v) \geq -\frac{\Lambda}{R^2} \left(|v|^2 + Ct \right) e^{\lambda t}, \quad \forall (t, v) \in [0, T] \times B_R.$$

We let $R \to +\infty$ and obtain $f^\varepsilon(t,v) \geq 0$ for every $(t,v) \in [0,T] \times \mathbb{R}^3$.

Next, we introduce the quasilinear operator $\mathcal{L}_2$ defined by

$$\mathcal{L}_2 u = \partial_t u - \sum_{i,j} \left(\overline{A}^{g,\varepsilon}_{i,j} + \varepsilon\, \delta_{i,j} \right) \partial^2_{i,j} u - \sum_i \left[\overline{B}^{g,\varepsilon}_i - \overline{b}^{g,\varepsilon}_i (1 - 2f^\varepsilon) \right] \partial_i u + \overline{c}^{g,\varepsilon} f^\varepsilon\, \theta(u).$$

Let $R > 0$. For

$$C \geq 6K_\varepsilon \|f_{in}\|_{L^1} + 6\varepsilon + 12\,(1+\Lambda)^2 K^2_\varepsilon \|f_{in}\|^2_{L^1},$$

it follows from the comparison principle for quasilinear equations [19, Theorem 9.1] that

$$f^\varepsilon(t,v) \leq 1 + \frac{\Lambda}{R^2} \left(|v|^2 + Ct \right) e^t, \qquad \forall (t,v) \in [0,T] \times B_R.$$

Letting R go to infinity, we obtain that $f^\varepsilon(t,v) \leq 1$ for every $(t,v) \in [0,T] \times \mathbb{R}^3$.

Consequently, there exists a solution to (4.3) in $\mathcal{H}^{2+\delta,\,(2+\delta)/2}([0,T] \times \mathbb{R}^3)$. The uniqueness of such a solution follows easily from the comparison principle [17, Theorem 1.2.5].

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