Well-posedness and large time behaviour for the non-cutoff Kac equation with a Gaussian thermostat

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Abstract

We consider here a Kac equation with a Gaussian thermostat in the case of a non-cutoff cross section. Under the sole assumptions of finite mass and finite energy for the initial data, we prove the existence of a global in time solution for which mass and energy are preserved. Then, via Fourier transform techniques, we show that this solution is smooth, unique and converges to the corresponding stationary state.

1 Introduction

We consider here an integro-differential equation referred to as the non-cutoff Kac equation with a Gaussian thermostat. This equation is obtained as the limit when $N \to +\infty$ of a system of N particles that undergo binary elastic collisions and that is subjected to a force field E. The positions of the particles are neglected and their velocities are assumed to be one-dimensional. If we only assumed that the particles were accelerated by a force field, the system would no more be conservative. Therefore, as it has been done in many fields of statistical physics and molecular dynamics (see e.g. [23, 19, 29, 30] and the references therein), heat is removed in order to achieve a stationary state. This is done by introducing a damping term whose aim is to model the interaction of the system with a ideal heat bath. This additional term is based on Gauss' principle of least constraint [18] and amounts to projecting the force field onto the tangent plane to the energy surface. Thereby, the kinetic energy remains constant. This construction is known as a Gaussian isokinetic thermostat. The influence of such thermostats has been widely analyzed for the Lorentz gas, as well from the microscopic point of view [8, 9, 4] as from the kinetic one [28, 3, 5] where only at most linear collision operators have been considered. We are interested here in the consequence of such a friction term on a kinetic equation with a nonlinear collision operator with a singular cross section. With the above assumption the distribution function f of the limit-system satisfies (see [35])

$$\partial_t f + E\, \partial_v((1 - \zeta_f(t)v)f) = Q(f, f), \qquad (1)$$

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where $\zeta_f(t) = \int_{\mathbb{R}} v f(t,v)\,dv$ and

$$Q(f,f)(t,v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} (f(t,v')f(t,v'_*) - f(t,v)f(t,v_*))\, b(\theta)\, d\theta\, dv_*,$$

with

$$v' = v\cos\theta - v_*\sin\theta, \qquad v'_* = v\sin\theta + v_*\cos\theta.$$

For $t \geq 0$ and $v \in \mathbb{R}$, $f(t,v) \geq 0$ represents the density of particles with velocity v at time t. When no confusion can occur, we use the notation ζ instead of ζ_f. In (1), the left-hand side comes from the construction of the themostated force field which was described above and the right-hand side corresponds to the Kac collision operator (see [24]). In the original model, the scattering angle θ was chosen uniformly in $[-\pi,\pi)$, which implied that the cross section b was given by $b(\theta) = 1/(2\pi)$. We consider here a generalization and assume that b satisfies

$$b(\theta) = |\theta|^{-1-\alpha}, \qquad \theta \in (-\pi,\pi), \qquad \alpha \in (0,2). \tag{2}$$

Such a cross section has already been introduced by Desvillettes [11] for the Kac equation with no force field. Equation (1) is supplemented with the initial condition

$$f(0) = f_{in}, \tag{3}$$

where

$$f_{in} \in L^1(\mathbb{R}), \qquad f_{in} \geq 0, \qquad \int_{\mathbb{R}} f_{in}(v)\,dv = 1 = \int_{\mathbb{R}} f_{in}(v)\,v^2\,dv. \tag{4}$$

Our purpose is to study the well-posedness of (1), (3) and the large time behaviour of its solutions. We first observe that (1) was obtained as the limit when $N \to +\infty$ of a N-particles system whose energy was constant and taken equal to one. This explains why we consider here initial conditions whose energy is one. On the other hand, we note that it is not restrictive to assume that the initial condition has mass one. Indeed, consider $\varrho > 0$ and g_{in} satisfying (4) except that the mass of g_{in} is ϱ. Then, the function f_{in} defined by

$$f_{in}(v) = \varrho^{-3/2} g_{in}\left(\frac{v}{\sqrt{\varrho}}\right), \qquad v \in \mathbb{R},$$

satisfies (4). If we denote by f a solution to (1), (3) where E is replaced with $E/\sqrt{\varrho}$, then the function g defined by

$$g(t,v) = \varrho^{3/2} f(\varrho t, \sqrt{\varrho} v), \qquad (t,v) \in (0,+\infty) \times \mathbb{R},$$

is a solution to (1) with initial condition g_{in}.

As for the Kac equation with no force field, the usual *a priori* estimates are available here, that is mass and energy are formally preserved. For every $t \geq 0$,

$$\int_{\mathbb{R}} f(t,v)\,dv = \int_{\mathbb{R}} f_{in}(v)\,dv = 1 = \int_{\mathbb{R}} f_{in}(v)\,v^2\,dv = \int_{\mathbb{R}} f(t,v)\,v^2\,dv. \tag{5}$$

However, it is not clear whether there exists an entropy for (1).

For the non-cutoff Kac equation with no force field, the mathematical theory is well-developed. The problem of existence, smoothness, uniqueness and positivity of solutions to this equation, as well as the convergence to the equilibrium have already been investigated (see [11, 21, 12, 32, 20]). The main difficulty in the analysis is the non-integrable singularity of the cross section b. Concerning the thermostated Kac equation with $b(\theta) = 1/(2\pi)$, the existence of solutions to both the stationary and the evolution problem has been proved and the large time behaviour has been studied (see [34, 35]). To our knowledge, the only works on the non-cutoff thermostated Kac equation concern the existence and uniqueness of stationary solutions [2] and numerical simulations for the evolution problem [31]. Therefore, our purpose is to investigate the existence, smoothness and uniqueness of a solution to (1), (3) and the convergence towards the associated stationary solution when time tends to $+\infty$. We first introduce some notations. We set $L^1_2(\mathbb{R}) = L^1(\mathbb{R}, (1+v^2)\,dv)$ and $\|g\|_p = \|g\|_{L^p(\mathbb{R})}$ for any $p \in \{1, \infty\}$ and $g \in L^p(\mathbb{R})$. Furthermore, we denote by $\mathcal{C}([0,+\infty); w - L^1(\mathbb{R}))$ the space of weakly continuous function in $L^1(\mathbb{R})$, that is the space of continuous function from $[0, +\infty)$ in $L^1(\mathbb{R})$ endowed with its weak topology. We now define the notion of weak solutions we consider here and state our main results.

Definition 1 *Consider f_{in} satisfying (4) and assume that b is given by (2). A nonnegative function $f \in \mathcal{C}([0,+\infty); w - L^1(\mathbb{R})) \cap L^\infty([0,+\infty); L^1_2(\mathbb{R}))$ is said to be a weak solution to (1), (3) if, for any $\psi \in \mathcal{C}^2_b(\mathbb{R})$ and $t \geq 0$, it satisfies*

$$\begin{aligned}\int_{\mathbb{R}} f(t,v)\,\psi(v)\,dv &= \int_{\mathbb{R}} f_{in}(v)\,\psi(v)\,dv + E\int_0^t\int_{\mathbb{R}}(1-\zeta(s)\,v)\,f(s,v)\,\psi'(v)\,dv\,ds \\ &+ \int_0^t\int_{\mathbb{R}^2} K^\psi(v,v_*)\,f(s,v)\,f(s,v_*)\,dv\,dv_*\,ds, \end{aligned} \tag{6}$$

where

$$K^\psi(v,v_*) = \int_{-\pi}^{\pi}(\psi(v') - \psi(v) + v_*\,\sin\theta\,\psi'(v))\,b(\theta)\,d\theta.$$

We note that, for $\psi \in \mathcal{C}^2_b(\mathbb{R})$, we have

$$\begin{aligned}\psi(v') - \psi(v) &= (v(\cos\theta - 1) - v_*\sin\theta)\psi'(v) \\ &+ (v(\cos\theta-1) - v_*\sin\theta)^2\int_0^1(1-u)\,\psi''(v + u(v'-v))\,du,\end{aligned}$$

so that,

$$\left|K^\psi(v,v_*)\right| \leq 2\,(1+v^2)\,\|\psi\|_{\mathcal{C}^2_b}\int_{-\pi}^{\pi}(1-\cos\theta)\,b(\theta)\,d\theta + v_*^2\,\|\psi\|_{\mathcal{C}^2_b}\int_{-\pi}^{\pi}\sin^2\theta\,b(\theta)\,d\theta.$$

Consequently, the last integral in (6) is well-defined. Moreover, if $\alpha \in (0,1)$, then $\theta b(\theta) \in L^1(-\pi,\pi)$ and

$$K^\psi(v,v_*) = \int_{-\pi}^{\pi}(\psi(v') - \psi(v))\,b(\theta)\,d\theta.$$

Theorem 1 *Assume that b is given by (2) and consider f_{in} satisfying (4). For all field strengths $E > 0$, there exists a weak solution f to (1), (3) such that (5) holds for every $t \geq 0$. Moreover, for every $\tau > 0$ and $\beta \in \mathbb{R}$,*

$$f \in L^\infty_{loc}([\tau,+\infty); H^\beta(\mathbb{R})). \tag{7}$$

Section 2 is devoted to the proof of the existence part which is performed in two steps: we first consider the case of an even nonnegative integrable cross section and the existence of a solution in that case is obtained by a fixed point argument. At first sight, the damping term induced by the thermostat seems to add nonlinearity in the equation and complicate the analysis. We thus begin with removing this nonlinearity by showing that the momentum ζ satisfies a differential equation which can be solved explicitly. Next, the existence of a solution when b satisfies (2) is obtained thanks to a truncation argument and weak compactness. Thanks to the de la Vallée Poussin theorem, we only assume here that the initial condition satisfies natural bounds, that is finite mass and energy.

For the Kac equation with no force field as for the Boltzmann equation, the non-cutoff collision operator is known to have a smoothing effect [12, 1, 13]. Once we have proved the existence of a solution to (1), (3), it is therefore natural to investigate the smoothness of this solution. For the cutoff thermostated Kac equation, the stationary solution may be either continuous or have a power-like singularity depending on the value of E [34] whereas for the non-cutoff thermostated Kac equation, the stationary solution is smooth for any $E > 0$ [2]. Also for the evolution problem, the solution is smooth whatever the value of E. The proof is reported in Section 3. It follows the same lines as in [12] for the Kac equation with no force field. It consists in showing, by induction, that the Fourier transform of the solution with respect to the velocity variable belongs to $L^\infty_{loc}([\tau,+\infty); L^2(\mathbb{R},(1+\xi^2)^\beta\, d\xi))$ for any $\tau > 0$ and $\beta \in \mathbb{R}$.

We then consider in Section 4 the question of uniqueness for solutions to (1), (3) and their large time behaviour. We use a method based on the Fourier transform and introduced in [22, 6, 32]. This method has already been adapted to prove the convergence towards stationary solutions for the thermostated Kac equation when $b \equiv 1/(2\pi)$ in [35]. However, since we assume here that the cross section b is not integrable, we can no more split the collision operator as the difference between a loss term and a gain term, and thus we can not directly perform the same manipulations as in [35]. Of course, we could consider solutions to truncated equations and then, by similar calculations and by passing to the limit, we would obtain an analogous result for the non-cutoff equation. But this result would only concern solutions that are obtained by truncation. Therefore, we work right away with the non-cutoff equation and we pay a particular attention to the singularity. Thereby, we prove some stability result stated in Proposition 9, from which we deduce both the uniqueness for solutions to (1), (3) and the convergence towards the associated stationary state. These results are presented below. For any function $f : \mathbb{R} \to \mathbb{R}$, we define its Fourier transform $\hat{f}$ by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-iv\xi} f(v)\, dv, \qquad \xi \in \mathbb{R},$$

and for $s > 0$, we consider the Fourier-based metric d_s given by

$$d_s(f,g) = \sup_{\xi\in\mathbb{R}} \frac{\hat{f}(\xi) - \hat{g}(\xi)}{|\xi|^s}$$

for any pair of probability measure f and g. For $s = m + r$, with $m \in \mathbb{N}$ and $0 < r \le 1$, if f and g are two probability measures with the same moments up to order m and finite moments of order s then $d_s(f,g)$ is finite [7]. In this paper, we only consider the case $s = 2$ and the case $s = 1$. We show that, as for the Boltzmann equation and the Kac equation for Maxwell molecules [32], the distance d_2 is nonexpansive along trajectories of the solutions to the thermostated Kac equation but it may occur only for large time. More precisely, we prove the following theorem.

Theorem 2 *Consider $E > 0$ and a cross section b satisfying (2). Let f_{in} and g_{in} be two functions satisfying (4) and*

$$\int_{\mathbb{R}} f_{in}(v)\, v\, dv = \int_{\mathbb{R}} g_{in}(v)\, v\, dv. \tag{8}$$

Denote by f and g two weak solutions to (1) with initial conditions respectively f_{in} and g_{in}. Then, there exists a function $J \in \mathcal{C}^\infty([0,+\infty))$ depending on E, $\int_{\mathbb{R}} f_{in}(v)\, v\, dv$ and $\int_{-\pi}^{\pi} b(\theta)(1-\cos\theta)\, d\theta$ satisfying $\lim_{t\to+\infty} J(t) = +\infty$, and such that, for every $t \geq 0$,

$$d_2(f(t,.), g(t,.)) \leq e^{-J(t)}\, d_2(f_{in}, g_{in}).$$

In the above theorem, $J(t)$ might be negative for small values of t, depending on the initial data. Then the probability metric d_2 would not be nonexpansive for small values of t. Nevertheless, if f and g are two weak solutions to (1) with the same initial condition, the previous theorem implies, by [15, Theorem 9.5.1], that, for every $t \geq 0$, the measure with density $f(t,.)$ is equal to the one with density $g(t,.)$.

Concerning the large time behaviour, we first recall that the associated stationary problem

$$E\, \frac{d}{dv}((1-\zeta v) f) = Q(f,f), \tag{9}$$

has already been considered in [2] where the following theorem has been established.

Theorem 3 *Assume that b satisfies (2). For all field strengths $E > 0$, there exists a unique weak solution f_{stat} to (9) such that*

$$\int_{\mathbb{R}} f_{stat}(v)\, dv = 1 = \int_{\mathbb{R}} f_{stat}(v)\, v^2\, dv.$$

and moments of any order of f_{stat} are finite. Moreover, $f_{stat} \in \mathcal{C}^\infty(\mathbb{R})$.

Whereas the exponential convergence towards equilibrium for the Kac or Boltzmann equation for Maxwell molecules has been proved in distance $d_{2+\delta}$ [22], we obtain here exponential convergence in distance d_1 thanks to the thermostat.

Theorem 4 *Consider $E > 0$ and a cross section b satisfying (2). Denote by f a weak solution to (1), (3), in the sense of Definition 1. Then, there exists a constant $C > 0$ such that*

$$d_1(f(t,.), f_{stat}) \leq C\, e^{-(\sqrt{K^2+4E^2}-K)t/2} + C\, e^{-t\sqrt{K^2+4E^2}}, \qquad t \geq 0,$$

where $K = \int_{-\pi}^{\pi} b(\theta)(1-\cos\theta)\, d\theta$.

This ensures, by [15, Theorem 9.8.2], that the measure with density $f(t,.)$ converges weakly-* to the measure with density f_{stat} when $t \to +\infty$.

2 Existence

2.1 Cutoff case

We consider here equations (1), (3) when $b : [-\pi, \pi] \to \mathbb{R}$ is an even nonnegative integrable function and prove the following theorem.

Theorem 5 *Consider f_{in} satisfying (4), and two nonnegative convex functions $\Phi_1, \Phi_2 \in \mathcal{C}^1([0, +\infty))$ such that $\Phi_i(0) = 0$, $\Phi_i'(0) = 0$, Φ_i' is concave, $\lim_{r\to+\infty} \Phi_i(r)/r = +\infty$ for $i \in \{1, 2\}$,*

$$\int_{\mathbb{R}} f_{in}(v)\, \Phi_1(v^2)\, dv < \infty \qquad \text{and} \qquad \int_{\mathbb{R}} \Phi_2(f_{in}(v))\, dv < \infty. \tag{10}$$

Let $b : [-\pi, \pi] \to \mathbb{R}$ be an even nonnegative and integrable function. For all field strengths $E > 0$, there exists a weak solution f to (1), (3) such that (5) holds and both $\int_{\mathbb{R}} f(., v)\, \Phi_1(v^2)\, dv$ and $\int_{\mathbb{R}} \Phi_2(f(., v))\, dv$ belong to $L^\infty_{loc}(0, +\infty)$.

We first notice that, if f is such a solution then ζ satisfies the following Cauchy problem

$$\zeta'(t) = E(1 - \zeta(t)^2) - K\zeta(t), \qquad t \geq 0, \tag{11}$$

$$\zeta(0) = \zeta_{in} = \int_{\mathbb{R}} f_{in}(v)\, v\, dv, \tag{12}$$

where $K = \int_{-\pi}^{\pi} b(\theta)(1-\cos\theta)\, d\theta$. This differential equation may be solved explicitly and the solution is given by

$$\overline{\zeta}(t) = \frac{\zeta_+(\zeta_- - \zeta_{in}) + \zeta_-(\zeta_{in} - \zeta_+)e^{-\sqrt{K^2+4E^2}t}}{\zeta_- - \zeta_{in} + (\zeta_{in} - \zeta_+)e^{-\sqrt{K^2+4E^2}t}}, \tag{13}$$

where

$$\zeta_\pm = \frac{-K \pm \sqrt{K^2 + 4E^2}}{2E}. \tag{14}$$

The existence of a solution to (1), (3) when $b : [-\pi, \pi] \to \mathbb{R}$ is an even nonnegative and integrable function is obtained by a fixed point argument. Let $M_1 \geq 4\, \Phi_1(1)$,

$$M_2 \geq \Phi_2'\left(\frac{1}{\sqrt{2\pi \min\{1 - \zeta_{in}^2, 1 - \zeta_+^2\}}}\right), \qquad L \geq \frac{4(E + K)}{(\min\{1 - \zeta_{in}^2, 1 - \zeta_+^2\})^2},$$

and T be four positive real numbers, the values of which will be specified later. We denote by $\mathcal{H}$ the set of nonnegative functions $h \in \mathcal{C}([0, T]; w - L^1(\mathbb{R}))$ such that, for every $s, t \in [0, T]$ and every $\psi \in \mathcal{C}_b^1(\mathbb{R})$,

$$\int_{\mathbb{R}} h(t, v)\, dv = 1 = \int_{\mathbb{R}} h(t, v)\, v^2\, dv, \qquad \int_{\mathbb{R}} h(t, v)\, v\, dv = \overline{\zeta}(t), \tag{15}$$

$$\int_{\mathbb{R}} h(t, v)\, \Phi_1(v^2)\, dv \leq M_1, \qquad \int_{\mathbb{R}} \Phi_2(h(t, v))\, dv \leq M_2, \tag{16}$$

and

$$\left|\int_{\mathbb{R}} h(t, v)\, \psi(v)\, dv - \int_{\mathbb{R}} h(s, v)\, \psi(v)\, dv\right| \leq L\, |t - s|\, \|\psi\|_{\mathcal{C}_b^1}. \tag{17}$$

The function

$$(t,v) \mapsto \frac{1}{\sqrt{2\pi(1-\overline{\zeta}(t)^2)}} \exp\left(-\frac{(v-\overline{\zeta}(t))^2}{2(1-\overline{\zeta}(t)^2)}\right)$$

belongs to $\mathcal{H}$. Consequently, $\mathcal{H}$ is non-empty. For $h \in \mathcal{H}$, we consider the following equation

$$\partial_t f + E\,(1-\overline{\zeta}(t)v)\partial_v f + (\|b\|_1 - E\overline{\zeta}(t))f = Q_+(h,h), \tag{18}$$

where

$$Q_+(h,h)(t,v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} h(t,v')\, h(t,v'_*)\, b(\theta)\, d\theta\, dv_*. \tag{19}$$

The existence of a solution to (18), (19) is obtained by the method of characteristics. We have the following proposition.

Proposition 6 *Consider f_{in} satisfying (4) and two nonnegative convex functions $\Phi_1, \Phi_2 \in \mathcal{C}^1([0,+\infty))$ such that $\Phi_i(0) = 0$, $\Phi_i'(0) = 0$, Φ_i' is concave for $i \in \{1,2\}$ and (10) holds.*

Let $h \in \mathcal{C}([0,T]; w - L^1(\mathbb{R}))$. Setting

$$f(t,v) = f_{in}(V(0;t,v))e^{-\|b\|_1 t + E\,Z(t)} + \int_0^t e^{-\|b\|_1(t-s)+E\,(Z(t)-Z(s))} Q_+(h,h)(s,V(s;t,v))\, ds, \tag{20}$$

where

$$Z(t) = \int_0^t \overline{\zeta}(u)\, du \qquad \text{and} \qquad V(s;t,v) = v\, e^{E\,(Z(t)-Z(s))} - E\int_s^t e^{E\,(Z(\sigma)-Z(s))}\, d\sigma, \tag{21}$$

then f is the unique weak solution to (18) with initial condition f_{in}.

Moreover, there exists constants M_1, M_2, L and T depending only on E, b, f_{in}, Φ_1 and Φ_2 such that for every $h \in \mathcal{H}$, the function f given by (20) also belongs to $\mathcal{H}$.

Proof. The first assertion of Proposition 6 is classical. We next compute the first moments of f when $h \in \mathcal{H}$. Let $t \in [0,T]$. The change of variables $v_* = V(s;t,v)$ is a $\mathcal{C}^1$-diffeomorphism for every $(s,t) \in [0,T]$ and we have $v = V(t;s,v_*)$ with

$$\partial_v V(t;s,v_*) = e^{-E\,(Z(t)-Z(s))}.$$

Consequently, for any measurable function $\Lambda : \mathbb{R} \to [0,+\infty)$ or for any measurable function $\Lambda : \mathbb{R} \to \mathbb{R}$ such that $f(t,.)\Lambda \in L^1(\mathbb{R})$, we get

$$\begin{aligned} \int_{\mathbb{R}} f(t,v)\, \Lambda(v)\, dv &= e^{-\|b\|_1 t} \int_{\mathbb{R}} f_{in}(v)\, \Lambda(V(t;0,v))\, dv \\ &+ \int_0^t e^{-\|b\|_1(t-s)} \int_{\mathbb{R}} Q_+(h,h)(s,v)\, \Lambda(V(t;s,v))\, dv\, ds. \end{aligned} \tag{22}$$

Moreover, it is easily checked that, for every $s \in [0,T]$,

$$\int_{\mathbb{R}} Q_+(h,h)(s,v)\, dv = \|b\|_1 = \int_{\mathbb{R}} Q_+(h,h)(s,v)\, v^2\, dv, \tag{23}$$

and
$$\int_{\mathbb{R}} Q_+(h,h)(s,v)\, v\, dv = \overline{\zeta}(s) \int_{-\pi}^{\pi} b(\theta)\, \cos(\theta)\, d\theta. \tag{24}$$

Taking $\Lambda \equiv 1$, $\Lambda(v) = v$ and $\Lambda(v) = v^2$ in (22), we deduce from (23) and (24) that

$$\begin{aligned}
\int_{\mathbb{R}} f(t,v)\, dv &= e^{-\|b\|_1 t} + \|b\|_1 \int_0^t e^{-\|b\|_1 (t-s)}\, ds = 1, \qquad (25)\\
\int_{\mathbb{R}} f(t,v)\, v\, dv &= \zeta_{in}\, e^{-\|b\|_1 t - E\, Z(t)} + E e^{-\|b\|_1 t} \int_0^t e^{-E\,(Z(t)-Z(s))} ds \\
&+ \int_{-\pi}^{\pi} b(\theta)\, \cos(\theta)\, d\theta \int_0^t e^{-\|b\|_1 (t-s) - E\,(Z(t)-Z(s))} \overline{\zeta}(s)\, ds \\
&+ E \|b\|_1 \int_0^t e^{-\|b\|_1 (t-s)} \int_s^t e^{-E\,(Z(t)-Z(\sigma))}\, d\sigma\, ds, \qquad (26)\\
\int_{\mathbb{R}} f(t,v)\, v^2\, dv &= e^{-\|b\|_1 t - 2E\, Z(t)} + 2\, E\, \zeta_{in}\, e^{-\|b\|_1 t - E\, Z(t)} \int_0^t e^{-E\,(Z(t)-Z(s))} ds \\
&+ 2E \int_{-\pi}^{\pi} b(\theta)\, \cos(\theta)\, d\theta \int_0^t e^{-\|b\|_1 (t-s) - E\,(Z(t)-Z(s))} \int_s^t e^{-E\,(Z(t)-Z(\sigma))}\, d\sigma\, \overline{\zeta}(s)\, ds \\
&+ E^2\, e^{-\|b\|_1 t} \left(\int_0^t e^{-E\,(Z(t)-Z(s))} ds \right)^2 + \|b\|_1 \int_0^t e^{-\|b\|_1 (t-s) - 2E\,(Z(t)-Z(s))}\, ds \\
&+ E^2 \|b\|_1 \int_0^t e^{-\|b\|_1 (t-s)} \left(\int_s^t e^{-E\,(Z(t)-Z(\sigma))}\, d\sigma \right)^2 ds. \qquad (27)
\end{aligned}$$

Integrating by parts the last term in (26), we get

$$\int_{\mathbb{R}} f(t,v)\, v\, dv = \zeta_{in}\, e^{-\|b\|_1 t - E\, Z(t)} + \int_0^t e^{-\|b\|_1 (t-s) - E\,(Z(t)-Z(s))} \left(E + \overline{\zeta}(s) \int_{-\pi}^{\pi} b(\theta)\, \cos(\theta)\, d\theta \right) ds.$$

Besides, $\overline{\zeta}$ satisfies (11), which implies that

$$\frac{d}{ds} \left(e^{\|b\|_1 s + E\, Z(s)} \overline{\zeta}(s) \right) = e^{\|b\|_1 s + E\, Z(s)} \left(E + \overline{\zeta}(s) \int_{-\pi}^{\pi} b(\theta)\, \cos(\theta)\, d\theta \right), \tag{28}$$

and thus,
$$\int_{\mathbb{R}} f(t,v)\, v\, dv = \overline{\zeta}(t). \tag{29}$$

Then, integrating by parts the last two terms of (27), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} f(t,v)\, v^2\, dv \\
&\quad = 1 + 2\, E\, \zeta_{in}\, e^{-\|b\|_1 t - E\, Z(t)} \int_0^t e^{-E\,(Z(t)-Z(s))} ds - 2E \int_0^t e^{-\|b\|_1 (t-s) - 2E\,(Z(t)-Z(s))} \overline{\zeta}(s)\, ds \\
&\quad + 2\, E \int_0^t e^{-\|b\|_1 (t-s) - E\,(Z(t)-Z(s))} \int_s^t e^{-E\,(Z(t)-Z(\sigma))}\, d\sigma \left(E + \overline{\zeta}(s) \int_{-\pi}^{\pi} b(\theta)\, \cos(\theta)\, d\theta \right) ds.
\end{aligned}$$

It then follows from the Fubini theorem and (28) that

$$\int_{\mathbb{R}} f(t,v)\, v^2\, dv = 1. \tag{30}$$

We now consider (22) with $\Lambda(v) = \Phi_1(v^2)$. By (21), we have

$$V(t;s,v)^2 \le 2v^2 e^{-2E\,(Z(t)-Z(s))} + 2E^2 \left(\int_s^t e^{-E\,(Z(t)-Z(\sigma))}\, d\sigma \right)^2 .$$

Consequently, the monotonicity of Φ_1, the convexity of Φ_1 and Lemma 10 lead to

$$\begin{aligned} \Phi_1(V(t;s,v)^2) &\le \frac{1}{2}\,\Phi_1\left(4v^2 e^{-2E\,(Z(t)-Z(s))}\right) + \frac{1}{2}\,\Phi_1\left(4E^2 \left(\int_s^t e^{-E\,(Z(t)-Z(\sigma))}\, d\sigma \right)^2 \right) \\ &\le a_1(s,t)\,\Phi_1(v^2) + a_2(s,t), \end{aligned}$$

where

$$a_1(s,t) := \frac{1}{2} + 8e^{-4E\,(Z(t)-Z(s))}, \qquad a_2(s,t) := \frac{1}{2}\,\Phi_1(4E^2)\left(1 + \left(\int_s^t e^{-E\,(Z(t)-Z(\sigma))}\, d\sigma \right)^4 \right).$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}} f(t,v)\,\Phi_1(v^2)\, dv &\le e^{-\|b\|_1 t} a_1(0,t) \int_{\mathbb{R}} f_{in}(v)\,\Phi_1(v^2)\, dv + e^{-\|b\|_1 t} a_2(0,t) \\ &+ \int_0^t e^{-\|b\|_1 (t-s)} \left(a_1(s,t) \int_{\mathbb{R}} \Phi_1(v^2)\, Q_+(h,h)(s,v)\, dv + \|b\|_1\, a_2(s,t) \right) ds. \end{aligned}$$

A change of variables leads to

$$\int_{\mathbb{R}} \Phi_1(v^2) Q_+(h,h)(s,v)\, dv = \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} b(\theta)\,\Phi_1(v'^2)\, h(s,v)\, h(s,v_*)\, d\theta\, dv\, dv_*.$$

But

$$v'^2 \le 2\,v^2 \cos^2\theta + 2\,v_*^2 \sin^2\theta \le 2\,v^2 + 2\,v_*^2,$$

which implies, with the monotonicity of Φ_1, the convexity of Φ_1 and Lemma 10, that

$$\Phi_1(v'^2) \le \frac{1}{2}\,\Phi_1(4v^2) + \frac{1}{2}\,\Phi_1(4v_*^2) \le 8\,\Phi_1(v^2) + 8\,\Phi_1(v_*^2).$$

Consequently,

$$\int_{\mathbb{R}} \Phi_1(v^2) Q_+(h,h)(s,v)\, dv \le 16\,\|b\|_1 \int_{\mathbb{R}} \Phi_1(v^2) h(s,v)\, dv.$$

By (16), we obtain, after integrations by parts, that

$$\begin{aligned} \int_{\mathbb{R}} f(t,v)\Phi_1(v^2)\, dv &\le e^{-\|b\|_1 t}\left(\frac{1}{2} + 8e^{-4E\,Z(t)}\right) \int_{\mathbb{R}} f_{in}(v)\,\Phi_1(v^2)\, dv + 8M_1\left(1 - e^{-\|b\|_1 t}\right) \\ &+ 128 M_1 \left(1 - e^{-\|b\|_1 t - 4E\,Z(t)} - 4E \int_0^t e^{-\|b\|_1 (t-s) - 4E\,(Z(t)-Z(s))}\overline{\zeta}(s)\, ds \right) \\ &+ \Phi_1(4E^2)\left(\frac{1}{2} + 2\int_0^t e^{-\|b\|_1(t-s) - E\,(Z(t)-Z(s))} \left(\int_s^t e^{-E\,(Z(t)-Z(\sigma))}\, d\sigma \right)^3 ds \right). \end{aligned}$$

It then follows from the bounds

$$e^{-\|b\|_1(t-s)} \leq 1 \qquad \text{and} \qquad |\overline{\zeta}(s)| \leq \max\{|\zeta_{in}|, |\zeta_+|\} \leq 1, \tag{31}$$

for every $s \in [0,t]$ that,

$$\begin{aligned}\int_{\mathbb{R}} f(t,v)\Phi_1(v^2)\,dv &\leq \left(\frac{1}{2} + 8e^{4ET}\right)\int_{\mathbb{R}} f_{in}(v)\,\Phi_1(v^2)\,dv + 8M_1\left(1 - e^{-\|b\|_1 T}\right)\\ &+ 128M_1\left(1 - e^{-\|b\|_1 T - 4ET} + 4E\int_0^t e^{4E(t-s)}\,ds\right)\\ &+ \Phi_1(4E^2)\left(\frac{1}{2} + 2\int_0^t e^{E(t-s)}\left(\int_s^t e^{E(t-\sigma)}\,d\sigma\right)^3 ds\right).\end{aligned}$$

We finally obtain

$$\begin{aligned}\int_{\mathbb{R}} f(t,v)\Phi_1(v^2)\,dv &\leq \left(\frac{1}{2} + 8e^{4ET}\right)\int_{\mathbb{R}} f_{in}(v)\,\Phi_1(v^2)\,dv + \frac{1}{2}\,\Phi_1(4E^2)\left(1 + \frac{1}{E^4}\left(e^{4ET}-1\right)\right)\\ &+ 8M_1\left(1 - e^{-\|b\|_1 T}\right) + 128M_1\left(1 - e^{-\|b\|_1 T - 4ET} + e^{4ET} - 1\right).\end{aligned} \tag{32}$$

We now consider the integral involving Φ_2. First, the convexity of Φ_2 and (20) imply that

$$\begin{aligned}\Phi_2(f(t,v)) &\leq \frac{1}{2}\,\Phi_2\left(2f_{in}(V(0;t,v))e^{-\|b\|_1 t + E\,Z(t)}\right)\\ &+ \frac{1}{2}\,\Phi_2\left(2\int_0^t e^{-\|b\|_1(t-s)+E\,(Z(t)-Z(s))}Q_+(h,h)(s,V(s;t,v))\,ds\right).\end{aligned}$$

We then deduce from the Jensen inequality and Lemma 10 that

$$\begin{aligned}\Phi_2(f(t,v)) &\leq \frac{1}{2}\,\max\left\{1, 4e^{-2\|b\|_1 t + 2E\,Z(t)}\right\}\,\Phi_2(f_{in}(V(0;t,v)))\\ &+ \frac{1}{2}\int_0^t \max\left\{1, 4\|b\|_1^2 t^2 e^{-2\|b\|_1(t-s)+2E\,(Z(t)-Z(s))}\right\}\,\Phi_2\left(\frac{Q_+(h,h)(s,V(s;t,v))}{\|b\|_1}\right)\frac{ds}{t}.\end{aligned} \tag{33}$$

Due to the change of variables $w = V(s;t,v)$, we have

$$\int_{\mathbb{R}} \Phi_2\left(\frac{Q_+(h,h)(s,V(s;t,v))}{\|b\|_1}\right)dv = \int_{\mathbb{R}} \Phi_2\left(\frac{Q_+(h,h)(s,w)}{\|b\|_1}\right)e^{-E\,(Z(t)-Z(s))}\,dw. \tag{34}$$

Besides, it follows from the Jensen inequality that

$$\int_{\mathbb{R}} \Phi_2\left(\frac{Q_+(h,h)(s,w)}{\|b\|_1}\right)dw \leq \int_{\mathbb{R}}\int_{-\pi}^{\pi} \Phi_2\left(\int_{\mathbb{R}} h(s,v')\,h(s,v'_*)\,dv_*\right)b(\theta)\,\frac{d\theta}{\|b\|_1}\,dv.$$

Setting

$$\mathcal{A}_1 = \left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right] \qquad \text{and} \qquad \mathcal{A}_2 = \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{3\pi}{4}\right], \tag{35}$$

we now split the integral in two parts

$$\int_{\mathbb{R}} \Phi_2\left(\frac{Q_+(h,h)(s,w)}{\|b\|_1}\right) dw \leq \int_{\mathbb{R}}\int_{\mathcal{A}_1} \Phi_2\left(\int_{\mathbb{R}} h(s,v')\,h(s,v'_*)\,dv_*\right) b(\theta)\,\frac{d\theta}{\|b\|_1}\,dv + \int_{\mathbb{R}}\int_{\mathcal{A}_2} \Phi_2\left(\int_{\mathbb{R}} h(s,v')\,h(s,v'_*)\,dv_*\right) b(\theta)\,\frac{d\theta}{\|b\|_1}\,dv.$$

Changing variables leads to

$$\int_{\mathbb{R}} \Phi_2\left(\frac{Q_+(h,h)(s,w)}{\|b\|_1}\right) dw \leq \int_{\mathbb{R}}\int_{\mathcal{A}_1} \Phi_2\left(\int_{\mathbb{R}} h\left(s,\frac{v-u\sin\theta}{\cos\theta}\right) h(s,u)\,\frac{du}{|\cos\theta|}\right) \frac{b(\theta)\,d\theta}{\|b\|_1}\,dv + \int_{\mathbb{R}}\int_{\mathcal{A}_2} \Phi_2\left(\int_{\mathbb{R}} h(s,u)\,h\left(s,\frac{v-u\cos\theta}{\sin\theta}\right) \frac{du}{|\sin\theta|}\right) \frac{b(\theta)\,d\theta}{\|b\|_1}\,dv.$$

Consequently, by the Jensen inequality, we have

$$\int_{\mathbb{R}} \Phi_2\left(\frac{Q_+(h,h)(s,w)}{\|b\|_1}\right) dw \leq \int_{\mathbb{R}}\int_{\mathcal{A}_1}\int_{\mathbb{R}} \Phi_2\left(h\left(s,\frac{v-u\sin\theta}{\cos\theta}\right)\frac{1}{|\cos\theta|}\right) h(s,u)\,du\,\frac{b(\theta)\,d\theta}{\|b\|_1}\,dv + \int_{\mathbb{R}}\int_{\mathcal{A}_2}\int_{\mathbb{R}} \Phi_2\left(h\left(s,\frac{v-u\cos\theta}{\sin\theta}\right)\frac{1}{|\sin\theta|}\right) h(s,u)\,du\,\frac{b(\theta)\,d\theta}{\|b\|_1}\,dv.$$

We now infer from the successive use of the Fubini Theorem, Lemma 10 and a change of variables that

$$\begin{aligned}\int_{\mathbb{R}} \Phi_2\left(\frac{Q_+(h,h)(s,w)}{\|b\|_1}\right) dw &\leq \int_{\mathbb{R}} h(s,u)\int_{\mathcal{A}_1}\frac{1}{\cos^2\theta}\int_{\mathbb{R}}\Phi_2\left(h\left(s,\frac{v-u\sin\theta}{\cos\theta}\right)\right) dv\,\frac{b(\theta)\,d\theta}{\|b\|_1}\,du \\ &+ \int_{\mathbb{R}} h(s,u)\int_{\mathcal{A}_2}\frac{1}{\sin^2\theta}\int_{\mathbb{R}}\Phi_2\left(h\left(s,\frac{v-u\cos\theta}{\sin\theta}\right)\right) dv\,\frac{b(\theta)\,d\theta}{\|b\|_1}\,du \\ &\leq \int_{\mathbb{R}} h(s,u)\int_{\mathcal{A}_1}\frac{b(\theta)}{\|b\|_1\,|\cos\theta|}\int_{\mathbb{R}}\Phi_2\left(h(s,w)\right) dw\,d\theta\,du \\ &+ \int_{\mathbb{R}} h(s,u)\int_{\mathcal{A}_2}\frac{b(\theta)}{\|b\|_1\,|\sin\theta|}\int_{\mathbb{R}}\Phi_2\left(h(s,w)\right) dw\,d\theta\,du.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}} \Phi_2\left(\frac{Q_+(h,h)(s,w)}{\|b\|_1}\right)\,dw \leq \sqrt{2}\int_{\mathbb{R}} h(s,u)\,du\int_{-\pi}^{\pi}\frac{b(\theta)}{\|b\|_1}\,d\theta\int_{\mathbb{R}}\Phi_2\left(h(s,w)\right)\,dw \leq M_2\sqrt{2}. \tag{36}$$

Gathering (33), (34) and (36), we finally obtain

$$\begin{aligned}\int_{\mathbb{R}} \Phi_2(f(t,v))\,dv &\leq \frac{1}{2}\,\max\left\{1, 4e^{-2\|b\|_1 t+2E\,Z(t)}\right\}\,e^{-E\,Z(t)}\int_{\mathbb{R}}\Phi_2(f_{in}(v))\,dv \\ &+ \frac{M_2}{\sqrt{2}}\int_0^t \max\left\{1, 4\|b\|_1^2 t^2 e^{-2\|b\|_1(t-s)+2E\,(Z(t)-Z(s))}\right\}\,e^{-E\,(Z(t)-Z(s))}\,\frac{ds}{t}.\end{aligned}$$

It then follows from (31) that

$$\begin{aligned}\int_{\mathbb{R}} \Phi_2(f(t,v))\,dv &\leq \frac{1}{2}\,\max\left\{1, 4\,e^{2\,ET}\right\}\,e^{ET}\int_{\mathbb{R}}\Phi_2(f_{in}(v))\,dv \\ &+ \frac{M_2}{\sqrt{2}}\,\max\left\{1, 4\,\|b\|_1^2\,T^2\,e^{2ET}\right\}\,\frac{e^{ET}-1}{ET}.\end{aligned} \tag{37}$$

We next turn to the Lipschitz property (17). Consider $\psi \in \mathcal{C}_b^1(\mathbb{R})$ and $t, \sigma \in [0,T]$. Since f is a weak solution to (18), we have

$$\int_{\mathbb{R}} f(t,v)\,\psi(v)\,dv - \int_{\mathbb{R}} f(\sigma,v)\,\psi(v)\,dv = E\int_\sigma^t \int_{\mathbb{R}} (1-\overline{\zeta}(s)v)\,f(s,v)\,\psi'(v)\,dv\,ds$$
$$- \|b\|_1 \int_\sigma^t \int_{\mathbb{R}} f(s,v)\,\psi(v)\,dv\,ds + \int_\sigma^t \int_{\mathbb{R}^2} h(s,v)\,h(s,v_*) \int_{-\pi}^{\pi} b(\theta)\,\psi(v')\,d\theta\,dv\,dv_*\,ds.$$

Consequently,

$$\begin{aligned}\left|\int_{\mathbb{R}} f(t,v)\,\psi(v)\,dv - \int_{\mathbb{R}} f(\sigma,v)\,\psi(v)\,dv\right| &\le E\,\|\psi\|_{\mathcal{C}_b^1} \left|\int_\sigma^t \left(1+|\overline{\zeta}(s)|\right)\,ds\right| + 2\,\|b\|_1\,\|\psi\|_{\mathcal{C}_b^1}|t-\sigma| \\ &\le 2\,(E+\|b\|_1)\,\|\psi\|_{\mathcal{C}_b^1}\,|t-\sigma|. \qquad (38)\end{aligned}$$

Finally, we put

$$\begin{aligned} M_1 &= \max\left\{4\,\Phi_1(1),\ 20\,\Phi_1(4E^2),\ 36\int_{\mathbb{R}} f_{in}(v)\,\Phi_1(v^2)\,dv\right\}, \\ M_2 &= \max\left\{\frac{25}{2}\int_R \Phi_2(f_{in}(v))\,dv,\ \Phi_2'\left(\frac{1}{\sqrt{2\pi\min\{1-\zeta_{in}^2, 1-\zeta_+^2\}}}\right)\right\}, \\ L &= \max\left\{2\,(E+\|b\|_1),\ \frac{4(E+K)}{(\min\{1-\zeta_{in}^2, 1-\zeta_+^2\})^2}\right\}, \\ T &= \min\left\{1,\ \frac{1}{E}\ln\left(\frac{1}{2\,\|b\|_1}\right),\ \frac{1}{\|b\|_1+4E}\ln\left(\frac{385}{384}\right),\ \frac{1}{4E}\ln\left(1+E^4\right)\right\}. \end{aligned}$$

It then readily follows from (25), (29), (30), (32), (37) and (38) that f belongs to $\mathcal{H}$. □

Proof of Theorem 5. Let M_1, M_2, L and T be the constants given by Proposition 6. We consider the map $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ defined by $\mathcal{T}(h) = f$ where f is given by (20). Let us check that $\mathcal{T}$ is continuous for the topology of $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$ and that $\mathcal{H}$ is a compact subset of $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$.

SEQUENTIAL CONTINUITY OF $\mathcal{T}$

Let $(h_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ that converges to h in $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$. It is clear that h belongs to $\mathcal{H}$. We set $f = \mathcal{T}(h)$ and $f_k = \mathcal{T}(h_k)$. Let $\varphi \in L^\infty(\mathbb{R})$. Then, the change of variables $v_* = V(s; t, v)$ leads to

$$\int_{\mathbb{R}} (f - f_k)(t,v)\,\varphi(v)\,dv = \int_0^t e^{-\|b\|_1(t-s)} \int_{\mathbb{R}} (Q_+(h,h) - Q_+(h_k,h_k))(s,v)\,\varphi(V(t;s,v))\,dv\,ds,$$

for any $t \in [0,T]$. Changing again variables, we then obtain

$$\int_{\mathbb{R}} (f-f_k)(t,v)\,\varphi(v)\,dv$$
$$= \int_0^t e^{-\|b\|_1(t-s)} \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} b(\theta)\,\varphi(V(t;s,v'))\,(h(s,v)\,h(s,v_*) - h_k(s,v)\,h_k(s,v_*))\,d\theta\,dv_*\,dv\,ds.$$

Let us denote by $(\varphi_l)_{l\in\mathbb{N}}$ a sequence in $\mathcal{D}(\mathbb{R})$ such that

$$\varphi_l \to \varphi \quad \text{a.e. in } \mathbb{R} \qquad \text{and} \qquad \|\varphi_l\|_\infty \le \|\varphi\|_\infty. \tag{39}$$

For $R > 0$, we deduce from (15) that

$$\begin{aligned}\left|\int_{\mathbb{R}} (f - f_k)(t,v)\,\varphi(v)\,dv\right| &\le \left|\int_0^t e^{-\|b\|_1(t-s)}\,G^{k,l,R}(t,s)\,ds\right| + \int_0^t e^{-\|b\|_1(t-s)} I_1^{l,R}(t,s)\,ds \\ &+ \int_0^t e^{-\|b\|_1(t-s)} I_2^{k,l,R}(t,s)\,ds + \frac{4}{R^2}\,\|\varphi\|_\infty,\end{aligned} \tag{40}$$

where

$$\begin{aligned} G^{k,l,R}(t,s) &= \int_{-R}^{R}\int_{-R}^{R}\int_{-\pi}^{\pi} b(\theta)\,\varphi_l(V(t;s,v'))\,(h(s,v)\,h(s,v_*) - h_k(s,v)\,h_k(s,v_*))\,d\theta\,dv_*\,dv,\\ I_1^{l,R}(t,s) &= \int_{-R}^{R}\int_{-R}^{R}\int_{-\pi}^{\pi} b(\theta)\,|(\varphi-\varphi_l)(V(t;s,v'))|\,h(s,v)\,h(s,v_*)\,d\theta\,dv_*\,dv,\\ I_2^{k,l,R}(t,s) &= \int_{-R}^{R}\int_{-R}^{R}\int_{-\pi}^{\pi} b(\theta)\,|(\varphi-\varphi_l)(V(t;s,v'))|\,h_k(s,v)\,h_k(s,v_*)\,d\theta\,dv_*\,dv.\end{aligned}$$

Since $(h_k)_{k\in\mathbb{N}}$ converges to h in $\mathcal{C}([0,T];w-L^1(\mathbb{R}))$, it follows by classical arguments that, for every $s\in[0,T]$,

$$h_k(s,v)\,h_k(s,v_*) \rightharpoonup h(s,v)\,h(s,v_*) \quad \text{weakly in } L^1((-R,R)^2).$$

For every $t,s\in[0,T]$, the mapping $(v,v_*)\mapsto \int_{-\pi}^{\pi} b(\theta)\,\varphi_l(V(t;s,v'))\,d\theta$ belongs to $L^\infty((-R,R)^2)$. Therefore, $G^{k,l,R}$ converges pointwise to 0 as $k\to+\infty$. Since $G^{k,l,R}$ is bounded by $2\,\|b\|_1\,\|\varphi\|_\infty$, the Lebesgue dominated convergence theorem implies that for every $t\in[0,T]$,

$$\lim_{k\to+\infty} \int_0^t e^{-\|b\|_1(t-s)}\,G^{k,l,R}(t,s)\,ds = 0.$$

But, the equicontinuity of the mapping $t\mapsto \int_0^t e^{-\|b\|_1(t-s)}\,G^{k,l,R}(t,s)\,ds$ enables us to conclude that this convergence is uniform on $[0,T]$. For every $l\in\mathbb{N}$ and $R>0$, we thus have

$$\lim_{k\to+\infty}\sup_{t\in[0,T]} \int_0^t e^{-\|b\|_1(t-s)}\,G^{k,l,R}(t,s)\,ds = 0. \tag{41}$$

On the other hand, for $\Upsilon > 0$,

$$I_1^{l,R}(t,s) \le \Upsilon^2 \int_{-R}^{R}\int_{-R}^{R}\int_{-\pi}^{\pi} b(\theta)\,|(\varphi-\varphi_l)(V(t;s,v'))|\,d\theta\,dv_*\,dv + 4\|b\|_1\|\varphi\|_\infty \int_{\mathbb{R}} \mathbf{1}_{h(s,v)\ge\Upsilon}\,h(s,v)\,dv.$$

Changing variables in the first integral and recalling that h satisfies (16), we obtain

$$I_1^{l,R}(t,s) \le 4R\|b\|_1\Upsilon^2 \int_{-2R}^{2R} |(\varphi-\varphi_l)(V(t;s,v))|\,dv + 4M_2\|b\|_1\|\varphi\|_\infty \sup_{r\ge\Upsilon} \frac{r}{\Phi_2(r)}.$$

For $s,t \in [0,T]$ and $v \in [-2R,2R]$, we have $|V(t;s,v)| \le (2R+ET)\, e^{ET} =: S$. Consequently, changing again variables, we get

$$I_1^{l,R}(t,s) \le 4R\|b\|_1 \Upsilon^2 e^{ET} \int_{-S}^{S} |(\varphi - \varphi_l)(u)|\, du + 4M_2 \|b\|_1 \|\varphi\|_\infty \sup_{r\ge\Upsilon} \frac{r}{\Phi_2(r)}. \tag{42}$$

Proceeding along the same lines for $I_2^{k,l,R}$, we obtain

$$I_2^{k,l,R}(t,s) \le 4R\|b\|_1 \Upsilon^2 e^{ET} \int_{-S}^{S} |(\varphi - \varphi_l)(u)|\, du + 4M_2 \|b\|_1 \|\varphi\|_\infty \sup_{r\ge\Upsilon} \frac{r}{\Phi_2(r)}. \tag{43}$$

Gathering (40), (41), (42) and (43), we deduce that

$$\begin{aligned} \limsup_{k\to+\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}} (f-f_k)(t,v)\, \varphi(v)\, dv \right| &\le 8R\Upsilon^2 e^{ET} \int_{-S}^{S} |(\varphi-\varphi_l)(u)|\, du \\ &+ 8M_2 \|\varphi\|_\infty \sup_{r\ge\Upsilon} \frac{r}{\Phi_2(r)} + \frac{4}{R^2}\, \|\varphi\|_\infty. \end{aligned}$$

We first let $l \to +\infty$ thanks to the Lebesgue dominated convergence theorem and then $R \to +\infty$, $\Upsilon \to +\infty$. We thus obtain that f_k converges to f in $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$, which means that $\mathcal{T}$ is sequentially continuous for the topology of $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$.

Compactness of $\mathcal{H}$

Due to the Arzela-Ascoli theorem [33, Theorem 1.3.2], it suffices to check that

$$\text{the family } \mathcal{H} \text{ is weakly equicontinuous,} \tag{44}$$

$$\text{the set } \{h(t), h \in \mathcal{H}\} \text{ is weakly relatively compact in } L^1(\mathbb{R}) \tag{45}$$

for every $t \in [0,T]$, to conclude that $\mathcal{H}$ is relatively compact in $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$. Let $t \in [0,T]$. It follows readily from the definition of $\mathcal{H}$ that

$$\sup_{h\in\mathcal{H}} \sup_{t\in[0,T]} \left\{ \int_{\mathbb{R}} h(t,v)\,(1+v^2)\, dv + \int_{\mathbb{R}} \Phi_2(h(t,v))\, dv \right\} < \infty, \tag{46}$$

whence (45) by the Dunford-Pettis theorem. Let us now consider (44). Let $\varphi \in L^\infty(\mathbb{R})$. There exists a sequence $(\varphi_l)_{l\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R})$ such that (39) holds. We fix $\kappa \in (0,1)$. We then deduce from (46) the existence of some real $\delta(\kappa) > 0$ such that, for any measurable subset E of $\mathbb{R}$ satisfying $\text{meas}(E) \le \delta(\kappa)$, we have

$$\sup_{h\in\mathcal{H}} \sup_{t\in[0,T]} \int_E h(t,v)\, dv \le \kappa. \tag{47}$$

Moreover, the Egorov theorem and (39) ensure the existence of a measurable subset E_κ of $[-1/\kappa, 1/\kappa]$ such that

$$\text{meas}(E_\kappa) \le \delta(\kappa) \qquad \text{and} \qquad \lim_{l\to+\infty} \sup_{[-1/\kappa,1/\kappa]\setminus E_\kappa} |\varphi_l - \varphi| = 0.$$

Consequently, for every $t \in (0,T)$, $s \in (-t, T-t)$ and $R \in [0, 1/\kappa]$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} (h(t+s,v) - h(t,v))\,\varphi(v)\,dv\right| &\leq \left|\int_{\mathbb{R}} (h(t+s,v) - h(t,v))\,\varphi_l(v)\,dv\right| \\ &\quad + \left|\int_{-R}^{R} (h(t+s,v) - h(t,v))\,(\varphi(v) - \varphi_l(v))\,dv\right| \\ &\quad + \left|\int_{|v|\geq R} (h(t+s,v) - h(t,v))\,(\varphi(v) - \varphi_l(v))\,dv\right| \\ &\leq L\,|s|\,\|\varphi_l\|_{\mathcal{C}_b^1} + 2\sup_{[-R,R]\setminus E_\kappa} |\varphi - \varphi_l| + 4\kappa\|\varphi\|_\infty + \frac{4\|\varphi\|_\infty}{R^2},\end{aligned}$$

by (15), (17), (39) and (47). Letting $s \to 0$, we thus obtain that, for every $t \in [0,T]$,

$$\limsup_{s\to 0}\sup_{h\in\mathcal{H}} \left|\int_{\mathbb{R}} (h(t+s,v) - h(t,v))\,\varphi(v)\,dv\right| \leq 2\sup_{[-R,R]\setminus E_\kappa} |\varphi - \varphi_l| + 4\kappa\|\varphi\|_\infty + \frac{4\|\varphi\|_\infty}{R^2}.$$

We now pass to the successive limit $l \to +\infty$, $\kappa \to 0$ and $R \to +\infty$ and deduce that (44) holds.

We are now in a position to complete the proof of Theorem 5. Indeed, following the same lines as in [17, Theorem 8.12.4], we may prove that if $F \subset \mathcal{H}$ is sequentially closed then F is closed. It then follows that $\mathcal{H}$ is a non-empty compact convex subset of $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$ and $\mathcal{T}$ is a continuous mapping from $\mathcal{H}$ to $\mathcal{H}$. The Tykhonov fixed point theorem [16, Theorem V.10.5] thus ensures the existence of a fixed point of $\mathcal{T}$, that is of a solution $f^1 \in \mathcal{C}([0,T]; w - L^1(\mathbb{R})) \cap L^\infty((0,T); L^1_2(\mathbb{R}))$ to (1), (3). Observing that T only depends on E and b, we may proceed as before with initial condition $f^1(T,.)$ instead of f_{in}. Repeating this argument, it finally yields the existence of a solution $f \in \mathcal{C}([0,+\infty); w - L^1(\mathbb{R})) \cap L^\infty((0,+\infty); L^1_2(\mathbb{R}))$ to (1), (3) that satisfies the desired properties. □

The next section is devoted to the non-cutoff case. We prove the existence of a weak solution to (1), (3) when b is given by (2). To this aim, we use a truncation argument. The weak solution to the non-cutoff equation is obtained as the limit as $n \to +\infty$ of a subsequence of solutions to (1), (3) with cross section $b_n := \min\{b, n\}$. We thus need uniform estimates, with respect to $n \in \mathbb{N}$, which is the purpose of the following two lemmas.

Lemma 7 *Let $f \in \mathcal{C}([0,+\infty); w - L^1(\mathbb{R})) \cap L^\infty((0,+\infty); L^1_2(\mathbb{R}))$ denote the weak solution given by Theorem 5. Then, for every $t \geq 0$, we have*

$$\int_{\mathbb{R}} f(t,v)\,\Phi_1(v^2)\,dv \leq e^{8(E+8\gamma_1)t}\int_{\mathbb{R}} f_{in}(v)\,\Phi_1(v^2)\,dv + E\Phi_1'(1)\,\frac{e^{8(E+8\gamma_1)t} - 1}{4(E+8\gamma_1)}, \tag{48}$$

where $\gamma_1 = \int_{-\pi}^{\pi} b(\theta)\,\sin^2\theta\,d\theta$.

Proof. The proof is inspired by that of [27, Lemma 2.2]. Let $t \in [0,+\infty)$. Since f is a weak solution to (1), (3), we have

$$\int_{\mathbb{R}} f(t,v)\,\Phi_1(v^2)\,dv = \int_{\mathbb{R}} f_{in}(v)\,\Phi_1(v^2)\,dv + 2E\int_0^t\int_{\mathbb{R}} (1 - \zeta(s)\,v)\,f(s,v)\,v\,\Phi_1'(v^2)\,dv\,ds \tag{49}$$

$$+ \frac{1}{2}\int_0^t\int_{\mathbb{R}^2} f(s,v)\,f(s,v_*)\,(G(v,v_*) - H(v,v_*))\,dv\,dv_*\,ds, \tag{50}$$

where

$$\begin{aligned} G(v,v_*) &= \int_{-\pi}^{\pi} b(\theta)\left(\Phi_1(v'^2)+\Phi_1({v'_*}^2)-\Phi_1(Y(\theta))-\Phi_1(Y(\pi/2-\theta))\right)\,d\theta,\\ H(v,v_*) &= \int_{-\pi}^{\pi} b(\theta)\left(\Phi_1(v^2)+\Phi_1(v_*^2)-\Phi_1(Y(\theta))-\Phi_1(Y(\pi/2-\theta))\right)\,d\theta, \end{aligned}$$

with $Y(\theta)=v^2\cos^2\theta+v_*^2\sin^2\theta$. We first consider G and H. The convexity of Φ_1 implies that

$$\begin{aligned} \Phi_1(Y(\theta)) &\leq \cos^2\theta\,\Phi_1(v^2)+\sin^2\theta\,\Phi_1(v_*^2),\\ \Phi_1(Y(\pi/2-\theta)) &\leq \sin^2\theta\,\Phi_1(v^2)+\cos^2\theta\,\Phi_1(v_*^2). \end{aligned}$$

It thus follows that $H(v,v_*)\geq 0$.

On the other hand, we infer from the convexity of Φ_1 that

$$\Phi_1(v'^2)-\Phi_1(Y(\theta))\geq \Phi_1'(Y(\theta))\,Z(\theta),$$

where $Z(\theta)=-v\,v_*\,\sin(2\theta)$. Since b is even, we then deduce that

$$\int_{-\pi}^{\pi} b(\theta)\,(\Phi_1(v'^2)-\Phi_1(Y(\theta)))\,d\theta\geq 0.$$

Similar calculations lead to

$$\int_{-\pi}^{\pi} b(\theta)\,(\Phi_1({v'_*}^2)-\Phi_1(Y(\pi/2-\theta)))\,d\theta\geq 0.$$

Consequently, we have $G(v,v_*)\geq 0$.

Lemma 11 implies that

$$\begin{aligned} \Phi_1(v'^2) &= (Y(\theta)+Z(\theta))\,\frac{\Phi_1(Y(\theta)+Z(\theta))}{Y(\theta)+Z(\theta)}\\ &\leq (Y(\theta)+Z(\theta))\,\left(\frac{\Phi_1(Y(\theta))}{Y(\theta)}+\frac{\Phi_1'(Y(\theta))\,Y(\theta)-\Phi_1(Y(\theta))}{Y(\theta)^2}\,Z(\theta)\right). \end{aligned}$$

Due to the nonnegativity of Φ_1, we finally obtain

$$\Phi_1(v'^2)-\Phi_1(Y(\theta))\leq\left(1+\frac{Z(\theta)}{Y(\theta)}\right)\,Z(\theta)\,\Phi_1'(Y(\theta)).$$

Multiplying the previous inequality by $b(\theta)$ and integrating with respect to θ leads to

$$\int_{-\pi}^{\pi} b(\theta)\,(\Phi_1(v'^2)-\Phi_1(Y(\theta)))\,d\theta\leq\int_{-\pi}^{\pi} b(\theta)\,\frac{Z(\theta)^2}{Y(\theta)}\,\Phi_1'(Y(\theta))\,d\theta.$$

Besides, for $\theta\in[-\pi,-3\pi/4]\cup[-\pi/4,\pi/4]\cup[3\pi/4,\pi]$, we get $Y(\theta)\geq v^2/2$ and thus

$$\frac{Z(\theta)^2}{Y(\theta)}\leq 8\,v_*^2\,\sin^2\theta\,\cos^2\theta.$$

On the other hand, for $\theta \in [-3\pi/4, -\pi/4] \cup [\pi/4, 3\pi/4]$, we get $Y(\theta) \geq v_*^2/2$ and thus

$$\frac{Z(\theta)^2}{Y(\theta)} \leq 8\, v^2 \sin^2\theta \cos^2\theta.$$

Therefore,

$$\int_{-\pi}^{\pi} b(\theta)\,(\Phi_1(v'^2) - \Phi_1(Y(\theta)))\,d\theta \leq 8 \int_{-\pi}^{\pi} b(\theta) \sin^2\theta \cos^2\theta\,(v^2 + v_*^2)\,\Phi_1'(Y(\theta))\,d\theta.$$

Owing to the concavity of Φ_1', we have

$$r\Phi_1'(r) \leq 2\,\Phi_1(r) \qquad \text{for every } r \geq 0, \tag{51}$$

by [25, Lemma A.1]. Then, the monotonicity of Φ_1', inequality (51), the convexity of Φ_1 and Lemma 10 entail that

$$(v^2 + v_*^2)\,\Phi_1'(Y(\theta)) \leq (v^2 + v_*^2)\,\Phi_1'(v^2 + v_*^2) \leq 4\,\Phi_1(v^2) + 4\,\Phi_1(v_*^2).$$

Therefore, we deduce that

$$\int_{-\pi}^{\pi} b(\theta)\,(\Phi_1(v'^2) - \Phi_1(Y(\theta)))\,d\theta \leq 32\,(\Phi_1(v^2) + \Phi_1(v_*^2)) \int_{-\pi}^{\pi} b(\theta) \sin^2\theta\,d\theta.$$

Similar calculations lead to

$$\int_{-\pi}^{\pi} b(\theta)\,(\Phi_1({v'_*}^2) - \Phi_1(Y(\pi/2 - \theta)))\,d\theta \leq 32\,(\Phi_1(v^2) + \Phi_1(v_*^2)) \int_{-\pi}^{\pi} b(\theta) \sin^2\theta\,d\theta.$$

We finally conclude that, for every $v, v_* \in \mathbb{R}$,

$$H(v, v_*) \geq 0 \qquad \text{and} \qquad 0 \leq G(v, v_*) \leq 64\,\gamma_1\,(\Phi_1(v^2) + \Phi_1(v_*^2)), \tag{52}$$

where $\gamma_1 = \int_{-\pi}^{\pi} b(\theta) \sin^2\theta\,d\theta$.

Let us now consider the last integral of (49). The nonnegativity of Φ_1', the monotonicity of Φ_1' and (51) imply that, for every $s \in [0, t]$,

$$\begin{aligned}
\left| \int_{\mathbb{R}} f(s,v)\, v\, \Phi_1'(v^2)\,dv \right| &\leq \int_{|v|\leq 1} f(s,v)\,\Phi_1'(v^2)\,dv + \int_{|v|\geq 1} f(s,v)\, v^2\, \Phi_1'(v^2)\,dv \\
&\leq \Phi_1'(1) \int_{|v|\leq 1} f(s,v)\,dv + 2 \int_{|v|\geq 1} f(s,v)\,\Phi_1(v^2)\,dv \\
&\leq \Phi_1'(1) + 2 \int_{\mathbb{R}} f(s,v)\,\Phi_1(v^2)\,dv. \qquad (53)
\end{aligned}$$

We then deduce from (51), (52), (53) and (31) that

$$\begin{aligned}
\int_{\mathbb{R}} f(t,v)\,\Phi_1(v^2)\,dv &\leq \int_{\mathbb{R}} f_{in}(v)\,\Phi_1(v^2)\,dv + 2\,E\,\Phi_1'(1)\,t + 8E \int_0^t \int_{\mathbb{R}} f(s,v)\,\Phi_1(v^2)\,dv\,ds \\
&+ 32\gamma_1 \int_0^t \int_{\mathbb{R}^2} f(s,v)\,f(s,v_*)\,(\Phi_1(v^2) + \Phi_1(v_*^2))\,dv\,dv_*\,ds.
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}} f(t,v)\,\Phi_1(v^2)\,dv \le \int_{\mathbb{R}} f_{in}(v)\,\Phi_1(v^2)\,dv + 2\,E\,\Phi_1'(1)\,t + 8\,(E+8\,\gamma_1)\int_0^t \int_{\mathbb{R}} f(s,v)\,\Phi_1(v^2)\,dv\,ds.$$

The Gronwall lemma then leads to the desired result. □

Lemma 8 *Let $f \in \mathcal{C}([0,+\infty); w-L^1(\mathbb{R})) \cap L^\infty((0,+\infty); L^1_2(\mathbb{R}))$ denote the weak solution given by Theorem 5. Then, for every $t \ge 0$, we have*

$$\int_{\mathbb{R}} \Phi_2(f(t,v))\,dv \le e^{\gamma_2\,t} \int_{\mathbb{R}} \Phi_2(f_{in}(v))\,dv,$$

with

$$\gamma_2 = E + \sqrt{2}\int_{\mathcal{A}_1} (1-\cos\theta)\,b(\theta)\,d\theta + \sqrt{2}\int_{\mathcal{A}_2} b(\theta)\,d\theta,$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined by (35).

Proof. Since f is a solution to (1), it satisfies, for every $t \ge 0$,

$$\begin{aligned}\int_{\mathbb{R}} \Phi_2(f(t,v))\,dv &= \int_{\mathbb{R}} \Phi_2(f_{in}(v))\,dv + E\int_0^t \zeta(s)\int_{\mathbb{R}} f(s,v)\,\Phi_2'(f(s,v))\,dv\,ds \\ &- E\int_0^t \zeta(s)\int_{\mathbb{R}} \Phi_2(f(s,v))\,dv\,ds + \int_0^t\int_{\mathbb{R}} Q(f,f)(s,v)\,\Phi_2'(f(s,v))\,dv\,ds. \quad (54)\end{aligned}$$

In order to justify (54), we use the following approximation arguments, in the spirit of [14]. Let $R > 0$. On the first hand, we introduce

$$\Phi_{2,R}(v) = \begin{cases} \Phi_2(v) & \text{if} \quad v \in [0,R], \\ \Phi_2'(R)(v-R) + \Phi_2(R) & \text{if} \quad v \in [R,+\infty) \end{cases}$$

Then, $\Phi_{2,R}$ is a Lipschitz function that belongs to $\mathcal{C}^1([0,+\infty))$ and satisfies $\Phi_{2,R} \le \Phi_2$. On the other hand, we consider

$$f_{in,\varepsilon}(v) = \int_{\mathbb{R}} f_{in}(u)\,\varrho_\varepsilon(v-u)\,du \qquad \text{and} \qquad \mathcal{Q}_{f,\varepsilon}(t,v) = \int_{\mathbb{R}} Q_+(f,f)(t,u)\,\varrho_\varepsilon(v-u)\,du,$$

where $\varrho_\varepsilon = \frac{1}{\varepsilon}\,\varrho(\frac{\cdot}{\varepsilon})$, $\varrho \in \mathcal{D}_+(\mathbb{R})$, $\varepsilon > 0$. Then, there exists a unique smooth solution f_ε to

$$\partial_t f_\varepsilon + E\,(1-\overline{\zeta}(t)v)\partial_v f_\varepsilon + (\|b\|_1 - E\overline{\zeta}(t))f_\varepsilon = \mathcal{Q}_{f,\varepsilon}, \qquad (55)$$

with initial condition $f_{in,\varepsilon}$ and this solution f_ε is given by (20) where f_{in} and $Q_+(h,h)$ are respectively replaced with $f_{in,\varepsilon}$ and $\mathcal{Q}_{f,\varepsilon}$. It then follows that f_ε converges to f in $\mathcal{C}([0,T]; L^1(\mathbb{R}))$ for any $T > 0$. Moreover, due to the smoothness of f_ε, we may now multiply (55) by $\Phi_{2,R}'(f_\varepsilon(t,v))$ and integrate by parts. Thereby, we obtain that f_ε satisfies, for every $t \ge 0$,

$$\begin{aligned}\int_{\mathbb{R}} \Phi_{2,R}(f_\varepsilon(t,v))\,dv &= \int_{\mathbb{R}} \Phi_{2,R}(f_{in,\varepsilon}(v))\,dv + \int_0^t (E\zeta(s) - \|b\|_1)\int_{\mathbb{R}} f_\varepsilon(s,v)\,\Phi_{2,R}'(f_\varepsilon(s,v))\,dv\,ds \\ &- E\int_0^t \zeta(s)\int_{\mathbb{R}} \Phi_{2,R}(f_\varepsilon(s,v))\,dv\,ds + \int_0^t\int_{\mathbb{R}} \mathcal{Q}_{f,\varepsilon}(s,v)\,\Phi_{2,R}'(f_\varepsilon(s,v))\,dv\,ds.\end{aligned}$$

We now pass to the successive limit $\varepsilon \to 0$ and $R \to +\infty$ and get that (54) holds.

Let us now show bounds for the second and the fourth integral in the right hand side of (54). By (51), we have

$$\int_{\mathbb{R}} f(s,v)\,\Phi_2'(f(s,v))\,dv \le 2\int_{\mathbb{R}} \Phi_2(f(s,v))\,dv. \tag{56}$$

Next,

$$\begin{aligned}\int_{\mathbb{R}} Q(f,f)(s,v)\,\Phi_2'(f(s,v))\,dv &= \int_{\mathbb{R}^2}\int_{\mathcal{A}_1} b(\theta)\,f(s,v)\,f(s,v_*)\,(\Phi_2'(f(s,v')) - \Phi_2'(f(s,v)))\,d\theta\,dv\,dv_* \\ &+ \int_{\mathbb{R}^2}\int_{\mathcal{A}_2} b(\theta)\,f(s,v)\,f(s,v_*)\,(\Phi_2'(f(s,v_*')) - \Phi_2'(f(s,v)))\,d\theta\,dv\,dv_*,\end{aligned}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given by (35). The convexity of Φ_2 entails that, for $x, y \ge 0$,

$$x(\Phi_2'(y) - \Phi_2'(x)) \le y\,\Phi_2'(y) - \Phi_2(y) + \Phi_2(x) - x\,\Phi_2'(x) \le \Psi(y) - \Psi(x),$$

where $\Psi(x) = x\,\Phi_2'(x) - \Phi_2(x)$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}} Q(f,f)(s,v)\,\Phi_2'(f(s,v))\,dv &\le \int_{\mathbb{R}^2}\int_{\mathcal{A}_1} b(\theta)\,f(s,v_*)\,(\Psi(f(s,v')) - \Psi(f(s,v)))\,d\theta\,dv\,dv_* \\ &+ \int_{\mathbb{R}^2}\int_{\mathcal{A}_2} b(\theta)\,f(s,v_*)\,(\Psi(f(s,v_*')) - \Psi(f(s,v)))\,d\theta\,dv\,dv_*. \end{aligned}\tag{57}$$

Changing variables leads to

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathcal{A}_1} b(\theta)\,f(s,v_*)\,(\Psi(f(s,v')) - \Psi(f(s,v)))\,d\theta\,dv\,dv_* \\ &= \int_{\mathbb{R}} f(s,v_*)\int_{\mathcal{A}_1} b(\theta)\int_{\mathbb{R}} \Psi(f(s,u))\frac{du}{|\cos\theta|}\,d\theta\,dv_* - \int_{\mathbb{R}} f(s,v_*)\int_{\mathcal{A}_1} b(\theta)\int_{\mathbb{R}} \Psi(f(s,v))\,dv\,d\theta\,dv_*.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^2}\int_{\mathcal{A}_1} b(\theta)\,f(s,v_*)\,(\Psi(f(s,v')) - \Psi(f(s,v)))\,d\theta\,dv\,dv_* = \int_{\mathbb{R}} \Psi(f(s,v))\,dv\int_{\mathcal{A}_1} b(\theta)\,\frac{1 - |\cos\theta|}{|\cos\theta|}\,d\theta. \tag{58}$$

Similarly,

$$\int_{\mathbb{R}^2}\int_{\mathcal{A}_2} b(\theta)\,f(s,v_*)\,(\Psi(f(s,v_*')) - \Psi(f(s,v)))\,d\theta\,dv\,dv_* = \int_{\mathbb{R}} \Psi(f(s,v))\,dv\int_{\mathcal{A}_2} b(\theta)\,\frac{1 - |\sin\theta|}{|\sin\theta|}\,d\theta. \tag{59}$$

It then readily follows from (54), (56), (57), (58) and (59) that

$$\begin{aligned}\int_{\mathbb{R}} \Phi_2(f(t,v))\,dv &\le \int_{\mathbb{R}} \Phi_2(f_{in}(v))\,dv + E\int_0^t\int_{\mathbb{R}} \Phi_2(f(s,v))\,dv\,ds \\ &+ \sqrt{2}\left(\int_{\mathcal{A}_1} b(\theta)\,(1-\cos\theta)\,d\theta + \int_{\mathcal{A}_2} b(\theta)\,d\theta\right)\int_0^t\int_{\mathbb{R}} \Psi(f(s,v))\,dv\,ds.\end{aligned}$$

By (51), we have $\Psi(y) \le \Phi_2(y)$ for every $y \ge 0$, which together with the Gronwall lemma lead to the desired result. □

2.2 Non-cutoff case

We now consider (1), (3) when the cross section b satisfies (2) and prove the existence part of Theorem 1. Since $|v|^2 \in L^1(\mathbb{R}, f_{in}(v)\,dv)$ and $f_{in} \in L^1(\mathbb{R})$, a refined version of the de la Vallée Poussin theorem [10, 26] ensures the existence of a function Φ_1 and a function Φ_2 fulfilling the assumptions of Theorem 5. Let $T > 0$. For $n \in \mathbb{N}$, we set $b_n := \min\{b, n\}$. Let $f_n \in \mathcal{C}([0,+\infty); w - L^1(\mathbb{R})) \cap L^\infty((0,+\infty); L^1_2(\mathbb{R}))$ be the weak solution to (1), (3) with the cross section b_n given by Theorem 5. We deduce from (5) and Lemma 8 that,

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \left\{ \int_{\mathbb{R}} f_n(t,v)\,(1+v^2)\,dv + \int_{\mathbb{R}} \Phi_2(f_n(t,v))\,dv \right\} < \infty.$$

The Dunford-Pettis theorem then ensures that, for every $t \in [0,T]$, $(f_n(t))_{n\in\mathbb{N}}$ is weakly relatively compact in $L^1(\mathbb{R})$. Due to the Arzela-Ascoli theorem [33, Theorem 1.3.2], it only remains to check that the family $f_n : [0,T] \to L^1(\mathbb{R})$ is weakly equicontinuous to conclude that the family $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$. Let $\psi \in \mathcal{D}(\mathbb{R})$. Since f_n is a weak solution to (1), (3), we have, for $t \in [0,T]$ and $s \in [-t, T-t]$,

$$\begin{aligned}\int_{\mathbb{R}} (f_n(t+s,v) - f_n(t,v))\,\psi(v)\,dv &= E \int_t^{t+s} \int_{\mathbb{R}} (1 - \zeta(\tau)v)\,f_n(\tau,v)\,\psi'(v)\,dv\,d\tau \\ &+ \int_t^{t+s} \int_{\mathbb{R}^2} f_n(\tau,v)\,f_n(\tau,v_*)\,K_n^\psi(v,v_*)\,dv\,dv_*\,d\tau, \end{aligned} \tag{60}$$

where

$$K_n^\psi(v,v_*) = \int_{-\pi}^{\pi} (\psi(v') - \psi(v) + v_*\,\sin\theta\,\psi'(v))\,b_n(\theta)\,d\theta.$$

For $(v,v_*) \in \mathbb{R}^2$,

$$\begin{aligned} K_n^\psi(v,v_*) &= \frac{1}{2}\,v_*^2 \int_{-\pi}^{\pi} \int_{-1}^{1} (1-|r|)\,\psi''(v\cos\theta + rv_*\sin\theta)\,dr\,\sin^2\theta\,b_n(\theta)\,d\theta \\ &- v \int_{-\pi}^{\pi} \int_0^1 \psi'(v + rv\,(\cos\theta - 1))\,dr\,(1-\cos\theta)\,b_n(\theta)\,d\theta. \end{aligned} \tag{61}$$

Consequently,

$$\left| K_n^\psi(v,v_*) \right| \le \|\psi\|_{\mathcal{C}_b^2} \left(\frac{1}{2}\,v_*^2 \int_{-\pi}^{\pi} b_n(\theta)\,\sin^2\theta\,d\theta + |v| \int_{-\pi}^{\pi} b_n(\theta)\,(1-\cos\theta)\,d\theta \right). \tag{62}$$

Since $b_n \le b$, we infer from (5), (31) and (62) that (60) reads

$$\left| \int_{\mathbb{R}} (f_n(t+s,v) - f_n(t,v))\,\psi(v)\,dv \right| \le |s|\,\|\psi\|_{\mathcal{C}_b^2} \left(2E + \frac{1}{2}\gamma_1 + K \right), \tag{63}$$

where

$$\gamma_1 = \int_{-\pi}^{\pi} b(\theta)\,\sin^2\theta\,d\theta \qquad \text{and} \qquad K = \int_{-\pi}^{\pi} b(\theta)\,(1-\cos\theta)\,d\theta. \tag{64}$$

Let $\varphi \in L^\infty(\mathbb{R})$. There exists a sequence $(\varphi_l)_{l\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R})$ such that (39) holds. Proceeding with (f_n) as we did with h to show the compactness of $\mathcal{H}$ in the proof of Theorem 5, we obtain, thanks to (63), that

$$\left| \int_{\mathbb{R}} (f_n(t+s,v) - f_n(t,v))\, \varphi(v)\, dv \right| \\ \leq |s|\, \|\varphi_l\|_{\mathcal{C}_b^2} \left(2E + \frac{1}{2}\gamma_1 + K\right) + 2 \sup_{[-R,R]\setminus E_\kappa} |\varphi - \varphi_l| + 4\,\kappa\, \|\varphi\|_\infty + \frac{4\,\|\varphi\|_\infty}{R^2},$$

where we kept the notations introduced in the proof of Theorem 5. Therefore, for every $t \in [0,T]$,

$$\limsup_{s\to 0} \sup_{n\geq 1} \left| \int_{\mathbb{R}} (f_n(t+s,v) - f_n(t,v))\, \varphi(v)\, dv \right| \leq 2 \sup_{[-R,R]\setminus E_\kappa} |\varphi - \varphi_l| + 4\,\kappa\, \|\varphi\|_\infty + \frac{4\,\|\varphi\|_\infty}{R^2}.$$

We now pass to the successive limit $l \to +\infty$, $\kappa \to 0$ and $R \to +\infty$ and deduce that the family $f_n : [0,T] \to L^1(\mathbb{R})$ is weakly equicontinuous. Consequently, there exist a nonnegative function f and a subsequence of $(f_n)_{n\in\mathbb{N}}$ (not relabelled) such that $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$. By Lemma 7, each f_n satisfies (48) and, passing to the limit $n \to +\infty$, we obtain that f also satisfies (48). It follows readily that (5) holds. It remains now to pass to the limit in (6). Consider $\psi \in \mathcal{C}_b^2(\mathbb{R})$, $t \in [0,T]$ and $R > 0$. It is straightforward that

$$\int_{\mathbb{R}} f_n(t,v)\, \psi(v)\, dv \to \int_{\mathbb{R}} f(t,v)\, \psi(v)\, dv,$$

when $n \to +\infty$. Next,

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} (1 - \zeta(s)\, v)\, (f_n - f)(s,v)\, \psi'(v)\, dv\, ds \right| \quad &\leq \quad \left| \int_0^t \int_{|v|\leq R} (1 - \zeta(s)\, v)\, (f_n - f)(s,v)\, \psi'(v)\, dv\, ds \right| \\ &+ \quad \int_0^t \int_{|v|\geq R} |1 - \zeta(s)\, v|\, (f_n + f)(s,v)\, |\psi'(v)|\, dv\, ds \end{aligned}$$

By (5), we thus have

$$\left| \int_0^t \int_{\mathbb{R}} (1 - \zeta(s)\, v)\, (f_n - f)(s,v)\, \psi'(v)\, dv\, ds \right| \\ \leq \left| \int_0^t \int_{|v|\leq R} (1 - \zeta(s)\, v)\, (f_n - f)(s,v)\, \psi'(v)\, dv\, ds \right| + \frac{2T(1+R)\|\psi\|_{\mathcal{C}_b^2}}{R^2}. \quad (65)$$

The convergence of f_n towards f in $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$ implies that the integral in the right hand side of (65) tends to 0 as $n \to +\infty$. Consequently,

$$\limsup_{n\to+\infty} \left| \int_0^t \int_{\mathbb{R}} (1 - \zeta(s)\, v)\, (f_n - f)(s,v)\, \psi'(v)\, dv\, ds \right| \leq \frac{2T(1+R)\|\psi\|_{\mathcal{C}_b^2}}{R^2}.$$

We may now let $R \to +\infty$. Let us turn our attention to the last integral of (6). We thus consider

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^2} K_n^\psi(v,v_*)\, f_n(s,v)\, f_n(s,v_*)\, dv\, dv_*\, ds - \int_0^t \int_{\mathbb{R}^2} K^\psi(v,v_*)\, f(s,v)\, f(s,v_*)\, dv\, dv_*\, ds \right| \\
&\qquad \le \left| \int_0^t \int_{\mathbb{R}^2} (K_n^\psi(v,v_*) - K^\psi(v,v_*))\, f_n(s,v)\, f_n(s,v_*)\, dv\, dv_*\, ds \right| \; (=: J_1) \\
&\qquad + \left| \int_0^t \int_{\mathbb{R}^2} K^\psi(v,v_*)\, (f_n(s,v)\, f_n(s,v_*) - f(s,v)\, f(s,v_*))\, dv\, dv_*\, ds \right|. \; (=: J_2)
\end{aligned}$$

It then follows from (61) that

$$\begin{aligned}
|\, K_n^\psi(v,v_*) - K^\psi(v,v_*)\, | \quad &\le \quad v_*^2\, \|\psi''\|_\infty \int_0^{n^{-1/(1+\alpha)}} \sin^2\theta\, b(\theta)\, d\theta \\
&+ \quad 2\,|v|\, \|\psi'\|_\infty \int_0^{n^{-1/(1+\alpha)}} (1-\cos\theta)\, b(\theta)\, d\theta,
\end{aligned}$$

which, together with (5) imply that J_1 tends to 0 as $n \to +\infty$. Next, we have

$$\begin{aligned}
J_2 \quad &\le \quad \left| \int_0^t \int_{-R}^R \int_{-R}^R K^\psi(v,v_*)\, (f_n(s,v_*)\, f_n(s,v) - f(s,v_*)\, f(s,v))\, dv_*\, dv\, ds \right| && (66)\\
&+ \quad \int_0^t \int_{\mathbb{R}} \int_{|v|\ge R} |K^\psi(v,v_*)|\, (f_n(s,v) f_n(s,v_*) + f(s,v) f(s,v_*))\, dv\, dv_*\, ds && (67)\\
&+ \quad \int_0^t \int_{\mathbb{R}} \int_{|v_*|\ge R} |K^\psi(v,v_*)|\, (f_n(s,v) f_n(s,v_*) + f(s,v) f(s,v_*))\, dv_*\, dv\, ds. && (68)
\end{aligned}$$

The convergence of (f_n) towards f in $\mathcal{C}([0,T]; w - L^1(\mathbb{R}))$ entails that

$$f_n(s,v)\, f_n(s,v_*) \to f(s,v)\, f(s,v_*) \qquad \text{in } \mathcal{C}([0,T]; w - L^1((-R,R)^2)).$$

Besides, since

$$|\, K^\psi(v,v_*)\, | \le \|\psi\|_{\mathcal{C}_b^2} \left(\frac{1}{2}\, \gamma_1\, v_*^2 + K\, |v| \right), \tag{69}$$

where γ_1 and K are defined by (64), we deduce that K^ψ belongs to $L^\infty((-R,R)^2)$. Consequently, (66) tends to 0 as n tends to $+\infty$. On the other hand, it follows from (5) and (69) that, for every $s \in [0,T]$,

$$\int_{\mathbb{R}} \int_{|v|\ge R} |K^\psi(v,v_*)|\, f_n(s,v)\, f_n(s,v_*)\, dv\, dv_* \le \|\psi\|_{\mathcal{C}_b^2} \left(\frac{\gamma_1}{2} + K \right) \frac{1}{R}. \tag{70}$$

A similar inequality holds if f_n is replaced with f. We now consider integral (68). Inequalities (69), (5) and (48) ensure that

$$\begin{aligned}
\int_{\mathbb{R}} \int_{|v_*|\ge R} |K^\psi(v,v_*)|\, f_n(s,v)\, f_n(s,v_*)\, dv_* dv \quad &\le \quad \|\psi\|_{\mathcal{C}_b^2} \left(\frac{\gamma_1}{2} \int_{|v_*|\ge R} v_*^2\, f_n(s,v_*)\, dv_* + \frac{K}{R^2} \right) \\
&\le \quad C\, \|\psi\|_{\mathcal{C}_b^2} \left(\sup_{|v_*|\ge R} \frac{v_*^2}{\Phi_1(v_*^2)} + \frac{1}{R^2} \right), \qquad (71)
\end{aligned}$$

where C only depends on E, b, f_{in}, Φ_1 and T. We obtain the same bound if f_n is replaced with f. Gathering (70) and (71), we conclude that

$$\limsup_{n\to\infty} J_2 \le T\,\|\psi\|_{\mathcal{C}_b^2}\,\frac{\gamma_1+2K}{R} + 2\,C\,T\,\|\psi\|_{\mathcal{C}_b^2}\left(\sup_{|v_*|\ge R}\frac{v_*^2}{\Phi_1(v_*^2)}+\frac{1}{R^2}\right).$$

We may now pass to the limit $R\to+\infty$ and we finally obtain that f satisfies (6).

3 Smoothness

We now turn our attention to the smoothness part of Theorem 1. Let f denote the weak solution to (1), (3) satisfying (5) obtained in Section 2.2. Let $T>0$ and $\tau\in(0,T)$. It follows readily from (5) that, for $\beta\le -1$, the Fourier transform $\hat f$ satisfies

$$\sup_{\tau\le t\le T}\int_{\mathbb{R}} |\hat f(t,\xi)|^2\,(1+\xi^2)^\beta\,d\xi<\infty, \tag{72}$$

which means that $f\in L^\infty([\tau,T];H^\beta(\mathbb{R}))$ for $\beta\le -1$. Let us prove that (72) also holds for $\beta>-1$. By (6), $\hat f$ satisfies

$$\partial_t \hat f(t,\xi) + iE\,\xi\,\hat f(t,\xi) + E\,\zeta_f(t)\,\xi\,\partial_\xi \hat f(t,\xi) = \hat Q(\hat f,\hat f)(t,\xi), \tag{73}$$

where

$$\hat Q(\hat f,\hat f)(t,\xi) = \int_{-\pi}^{\pi} b(\theta)\,\left(\hat f(t,\xi\cos\theta)\,\hat f(t,\xi\sin\theta) - \hat f(t,\xi) + i\sin\theta\,\zeta_f(t)\,\xi\,\hat f(t,\xi)\right)\,d\theta.$$

The change of variables $\theta\mapsto-\theta$ leads to the following equivalent form for the Fourier transform of the collision operator

$$\hat Q(\hat f,\hat f)(t,\xi) = \int_0^{\pi} b(\theta)\,\left(\hat f(t,\xi\cos\theta)\,\left(\hat f(t,\xi\sin\theta)+\hat f(t,-\xi\sin\theta)\right) - 2\hat f(t,\xi)\right)\,d\theta.$$

Let $\eta\in\mathcal{C}^\infty(\mathbb{R})$ be an even nonnegative function such that $\eta(v)=1$ if $|v|\le1$, $\eta(v)=0$ if $|v|\ge2$, $0\le\eta(v)\le1$ for every $v\in\mathbb{R}$ and η is nonincreasing on $\mathbb{R}_+$. For $R>0$, we define a function η_R by $\eta_R(v)=\eta(v/R)$ for every $v\in\mathbb{R}$. Let $\beta\in\mathbb{R}$. We now multiply (73) by $\overline{\hat f(t,\xi)}\,(1+\xi^2)^\beta\,\eta_R(\xi)$ and integrate with respect to the ξ variable. Taking the real part of the obtained equation, we get, thanks to the derivation under the integral sign theorem and an integration by parts that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}|\hat f(t,\xi)|^2\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi \;&=\; E\,\zeta_f(t)\int_{\mathbb{R}}|\hat f(t,\xi)|^2\,(1+\xi^2)^\beta\,\xi\,\eta_R'(\xi)\,d\xi\\ &+\; E\,\zeta_f(t)\int_{\mathbb{R}}|\hat f(t,\xi)|^2\left((1+\xi^2)^\beta+2\beta\xi^2\,(1+\xi^2)^{\beta-1}\right)\eta_R(\xi)\,d\xi\\ &+\; 2\int_{\mathbb{R}}\hat Q(\hat f,\hat f)(t,\xi)\,\overline{\hat f(t,\xi)}\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi. \end{aligned}\tag{74}$$

We notice that the change of variable $\xi\mapsto-\xi$ in the last term of (74) enables to prove that this integral is real. We now split this integral in the following way

$$2\int_{\mathbb{R}}\hat Q(\hat f,\hat f)(t,\xi)\,\overline{\hat f(t,\xi)}\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi = A_1+A_2+A_3, \tag{75}$$

where

$$
\begin{aligned}
A_1 &= \int_{\mathbb{R}}\int_0^{\pi/4} \Big(\Big(\hat f(\xi)\overline{\hat f(\xi\cos\theta)} + \overline{\hat f(\xi)}\hat f(\xi\cos\theta)\Big) \Big(\hat f(\xi\sin\theta) + \hat f(-\xi\sin\theta)\Big) \\
&- 2\left(|\hat f(\xi)|^2 + |\hat f(\xi\cos\theta)|^2\right)\Big)\, b(\theta)\,d\theta\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi, \\
A_2 &= 2\int_{\mathbb{R}}\int_0^{\pi/4} \left(|\hat f(\xi\cos\theta)|^2 - |\hat f(\xi)|^2\right) b(\theta)\,d\theta\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi, \\
A_3 &= 2\int_{\mathbb{R}}\int_{\pi/4}^{\pi} \Big(\hat f(\xi\cos\theta)\,(\hat f(\xi\sin\theta)+\hat f(-\xi\sin\theta)) - 2\hat f(\xi)\Big)\,\overline{\hat f(\xi)}\,(1+\xi^2)^\beta\,\eta_R(\xi)\,b(\theta)\,d\theta\,d\xi,
\end{aligned}
$$

and show bounds for each term. In the above formulas and in the following calculations, we omit the dependence on the t variable in order to simplify expressions. It follows from the inequality

$$
\left|\hat f(\xi)\overline{\hat f(\xi\cos\theta)} + \overline{\hat f(\xi)}\hat f(\xi\cos\theta)\right| \le |\hat f(\xi)|^2 + |\hat f(\xi\cos\theta)|^2,
$$

that

$$
\begin{aligned}
A_1 &\le -\int_{\mathbb{R}}\int_0^{\pi/4} \left(|\hat f(\xi)|^2 + |\hat f(\xi\cos\theta)|^2\right)\left(2 - \left|\hat f(\xi\sin\theta) + \hat f(-\xi\sin\theta)\right|\right) b(\theta)\,d\theta\,(1+\xi^2)^\beta \eta_R(\xi)\,d\xi \\
&\le -\int_{\mathbb{R}}\int_0^{\pi/4} |\hat f(\xi)|^2 \left(2 - \left|\hat f(\xi\sin\theta) + \hat f(-\xi\sin\theta)\right|\right) b(\theta)\,d\theta\,(1+\xi^2)^\beta \eta_R(\xi)\,d\xi.
\end{aligned}
$$

Now,

$$
2 - \left|\hat f(\xi\sin\theta) + \hat f(-\xi\sin\theta)\right| \ge 2\int_{\mathbb{R}} f(v)\,(1 - |\cos(\xi v\sin\theta)|)\,dv,
$$

which implies that

$$
A_1 \le -2\int_{\mathbb{R}}\int_0^{\pi/4} |\hat f(\xi)|^2 \int_{\mathbb{R}} f(v)\,(1-|\cos(\xi v\sin\theta)|)\,dv\,b(\theta)\,d\theta\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi.
$$

The change of variable $u = \xi v\sin\theta$ then leads to

$$
A_1 \le -2\int_{\mathbb{R}} |\hat f(\xi)|^2 \int_{\mathbb{R}} f(v) \int_0^{|\xi v|/\sqrt{2}} (1-|\cos u|)\left|\arcsin\left(\frac{u}{\xi v}\right)\right|^{-1-\alpha} \frac{du\,dv}{\sqrt{\xi^2 v^2 - u^2}}\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi.
$$

Since $\sqrt{\xi^2v^2-u^2} \le |\xi v|$ and $\arcsin(x) \le 2x$ for $x \in [0,1]$, we deduce that

$$
A_1 \le -2^{-\alpha}\int_{\mathbb{R}} |\hat f(\xi)|^2 \int_{\mathbb{R}} f(v)|v|^\alpha \int_0^{|\xi v|/\sqrt{2}} (1-|\cos u|)\,u^{-1-\alpha}\,du\,dv\,|\xi|^\alpha(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi.
$$

Consequently,

$$
A_1 \le -2^{-\alpha}\int_0^1 (1-|\cos u|)\,u^{-1-\alpha}\,du \int_{\mathbb{R}} f(v)|v|^\alpha \int_{|\xi|\ge\sqrt{2}/|v|} |\hat f(\xi)|^2\,|\xi|^\alpha(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi\,dv.
$$

Thus, we have

$$A_1 \leq -2^{-\alpha} \int_0^1 (1 - |\cos u|)\, u^{-1-\alpha}\, du \bigg(\int_{\mathbb{R}} f(v)|v|^\alpha\, dv \int_{\mathbb{R}} |\hat{f}(\xi)|^2\, |\xi|^\alpha (1+\xi^2)^\beta\, \eta_R(\xi)\, d\xi \\ - 2^{\alpha/2} \int_{\mathbb{R}} f(v) \int_{|\xi| \leq \sqrt{2}/|v|} |\hat{f}(\xi)|^2\, (1+\xi^2)^\beta\, \eta_R(\xi)\, d\xi \bigg) dv.$$

Let us recall at this point that f depends on t and we need to bound $\int_{\mathbb{R}} f(t,v)|v|^\alpha dv$ from below by a constant for $t \in [0,T]$. Let $\delta > 0$. Since f satisfies (5) and (48), there exists a constant K_T depending only on Φ_1, f_{in}, α, E and T such that

$$\begin{aligned} \int_{\mathbb{R}} f(t,v)|v|^\alpha dv &\geq \delta^{\alpha-2} \int_{|v|\leq\delta} f(t,v)\, v^2\, dv \\ &\geq \delta^{\alpha-2} \left(\int_{\mathbb{R}} f(t,v)\, v^2\, dv - \sup_{|u|\geq\delta} \frac{u^2}{\Phi_1(u^2)} \int_{|v|\geq\delta} f(t,v)\, \Phi_1(v^2)\, dv \right) \\ &\geq \delta^{\alpha-2} \left(1 - K_T \sup_{|u|\geq\delta} \frac{u^2}{\Phi_1(u^2)} \right). \end{aligned}$$

Since $\lim_{r\to+\infty} \Phi_1(r)/r = +\infty$, we infer that there exists $\delta > 0$ such that

$$1 - K_T \sup_{|u|\geq\delta} \frac{u^2}{\Phi_1(u^2)} \geq \frac{1}{2}.$$

Finally, there exist some constant $D_T > 0$ depending only on Φ_1, f_{in}, α, E, T and some constant $G > 0$ depending only on α such that

$$A_1 \leq -D_T \int_{\mathbb{R}} |\hat{f}(\xi)|^2\, |\xi|^\alpha (1+\xi^2)^\beta\, \eta_R(\xi)\, d\xi + G \int_{\mathbb{R}} |\hat{f}(\xi)|^2\, (1+\xi^2)^\beta\, \eta_R(\xi)\, d\xi. \tag{76}$$

We now consider A_2. The change of variable $u = \xi \cos\theta$ leads to

$$\begin{aligned} |A_2| &\leq 2 \int_0^{\pi/4} b(\theta) \left| \int_{\mathbb{R}} |\hat{f}(u)|^2 \left(1 + \frac{u^2}{\cos^2\theta}\right)^\beta \eta_R\left(\frac{u}{\cos\theta}\right) \frac{du}{|\cos\theta|} - \int_{\mathbb{R}} |\hat{f}(\xi)|^2\, (1+\xi^2)^\beta \eta_R(\xi)\, d\xi \right| d\theta \\ &\leq 2 \int_0^{\pi/4} b(\theta) \int_{\mathbb{R}} |\hat{f}(u)|^2 \left(1 + \frac{u^2}{\cos^2\theta}\right)^\beta \eta_R\left(\frac{u}{\cos\theta}\right) \left(\frac{1}{|\cos\theta|} - 1\right) du\, d\theta \\ &+ 2 \int_0^{\pi/4} b(\theta) \int_{\mathbb{R}} |\hat{f}(u)|^2 \left| \left(1 + \frac{u^2}{\cos^2\theta}\right)^\beta - (1+u^2)^\beta \right| \eta_R\left(\frac{u}{\cos\theta}\right) du\, d\theta \\ &+ 2 \int_0^{\pi/4} b(\theta) \int_{\mathbb{R}} |\hat{f}(u)|^2 (1+u^2)^\beta \left| \eta_R\left(\frac{u}{\cos\theta}\right) - \eta_R(u) \right| du\, d\theta. \end{aligned}$$

The Taylor formula gives

$$\left(1+\frac{u^2}{\cos^2\theta}\right)^\beta-(1+u^2)^\beta=\frac{2\,\beta\,(1-\cos\theta)}{\cos\theta}$$
$$\times\int_0^1 u^2\left(1+\frac{(\sigma-1)(\cos\theta-1)}{\cos\theta}\right)\left(1+u^2\left(1+\frac{(\sigma-1)(\cos\theta-1)}{\cos\theta}\right)^2\right)^{\beta-1}d\sigma,$$
$$\eta_R\left(\frac{u}{\cos\theta}\right)-\eta_R(u)=\left(\frac{1}{\cos\theta}-1\right)\int_0^1 u\,\eta_R'\left(u\left(1+(\sigma-1)\left(1-\frac{1}{\cos\theta}\right)\right)\right)d\sigma.$$

Therefore, there exists some constant C_β depending only on β and $\sup_{u\in\mathbb{R}}|u\,\eta'(u)|$ such that, for every $\theta\in[0,\pi/4]$ and $u\in\mathbb{R}$, we have

$$\begin{aligned}\left(1+\frac{u^2}{\cos^2\theta}\right)^\beta-(1+u^2)^\beta &\le C_\beta\left(\frac{1}{\cos\theta}-1\right)(1+u^2)^\beta,\\ \eta_R\left(\frac{u}{\cos\theta}\right)-\eta_R(u) &\le C_\beta\left(\frac{1}{\cos\theta}-1\right)\mathbf{1}_{R/\sqrt{2}\le|u|\le 2R},\end{aligned}$$

and

$$\left(1+\frac{u^2}{\cos^2\theta}\right)^\beta\le C_\beta\,(1+u^2)^\beta. \tag{77}$$

Moreover, since η_R is even and nonincreasing on $\mathbb{R}_+$, for every $\theta\in[0,\pi/4]$ and $u\in\mathbb{R}$,

$$\eta_R\left(\frac{u}{\cos\theta}\right)\le\eta_R(u). \tag{78}$$

We thus deduce that

$$\begin{aligned}|A_2| &\le 4\,C_\beta\int_0^{\pi/4}b(\theta)\left(\frac{1}{\cos\theta}-1\right)d\theta\int_{\mathbb{R}}|\hat f(u)|^2(1+u^2)^\beta\,\eta_R(u)\,du\\ &+ 2\,C_\beta\int_0^{\pi/4}b(\theta)\left(\frac{1}{\cos\theta}-1\right)d\theta\int_{\mathbb{R}}|\hat f(u)|^2(1+u^2)^\beta\,\mathbf{1}_{R/\sqrt{2}\le|u|\le 2R}\,du.\end{aligned} \tag{79}$$

Let us now turn our attention to A_3. By (5), we have

$$\begin{aligned}|A_3| &\le 4\int_{\mathbb{R}}\int_{\pi/4}^{3\pi/4}\left(|\hat f(\xi\sin\theta)||\hat f(\xi)|+|\hat f(\xi)|^2\right)(1+\xi^2)^\beta\,\eta_R(\xi)\,b(\theta)\,d\theta\,d\xi\\ &+ 4\int_{\mathbb{R}}\int_{3\pi/4}^{\pi}\left(|\hat f(\xi\cos\theta)||\hat f(\xi)|+|\hat f(\xi)|^2\right)(1+\xi^2)^\beta\,\eta_R(\xi)\,b(\theta)\,d\theta\,d\xi.\end{aligned}$$

Therefore,

$$\begin{aligned}|A_3| &\le 6\int_{\pi/4}^{\pi}b(\theta)\,d\theta\int_{\mathbb{R}}|\hat f(\xi)|^2\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi\\ &+ 2\int_{\pi/4}^{3\pi/4}\int_{\mathbb{R}}|\hat f(\xi\sin\theta)|^2\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi\,b(\theta)\,d\theta\\ &+ 2\int_{3\pi/4}^{\pi}\int_{\mathbb{R}}|\hat f(\xi\cos\theta)|^2\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi\,b(\theta)\,d\theta.\end{aligned}$$

The changes of variables $u = \xi \sin\theta$ and $u = \xi\cos\theta$ in the last two integrals lead to

$$\begin{aligned}|A_3| &\le 6\int_{\pi/4}^{\pi} b(\theta)\,d\theta \int_{\mathbb{R}} |\hat f(\xi)|^2\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi \\ &+ 2\int_{\pi/4}^{3\pi/4}\int_{\mathbb{R}} |\hat f(u)|^2\,\left(1+\frac{u^2}{\sin^2\theta}\right)^\beta\,\eta_R\left(\frac{u}{\sin\theta}\right)\,\frac{du}{|\sin\theta|}\,b(\theta)\,d\theta \\ &+ 2\int_{3\pi/4}^{\pi}\int_{\mathbb{R}} |\hat f(u)|^2\,\left(1+\frac{u^2}{\cos^2\theta}\right)^\beta\,\eta_R\left(\frac{u}{\cos\theta}\right)\,\frac{du}{|\cos\theta|}\,b(\theta)\,d\theta.\end{aligned}$$

Finally, arguing as for (77) and (78), we deduce that

$$|A_3| \le (6+2\sqrt{2}\,C_\beta)\int_{\pi/4}^{\pi} b(\theta)\,d\theta\int_{\mathbb{R}} |\hat f(\xi)|^2\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi. \tag{80}$$

We may now deduce from (74), (75), (76), (79) and (80) that there exists some constants $F_\beta, H_\beta > 0$ depending only on α, β, E, $\sup_{u\in\mathbb{R}} |u\,\eta'(u)|$ such that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} |\hat f(t,\xi)|^2\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi + D_T\int_{\mathbb{R}} |\hat f(t,\xi)|^2\,|\xi|^\alpha(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi \\ &\qquad\le F_\beta\int_{\mathbb{R}} |\hat f(t,\xi)|^2\,(1+\xi^2)^\beta\,\eta_R(\xi)\,d\xi + H_\beta\int_{\mathbb{R}} |\hat f(t,\xi)|^2\,(1+\xi^2)^\beta\,\mathbf{1}_{R/\sqrt{2}\le|\xi|\le 2R}\,d\xi.\end{aligned} \tag{81}$$

Taking $\beta = -1$ in (81), integrating and letting $R \to +\infty$, we infer, thanks to (5) that

$$\int_0^T\int_{\mathbb{R}} |\hat f(t,\xi)|^2\,|\xi|^\alpha(1+\xi^2)^{-1}\,d\xi\,dt \le \frac{1+T\,F_\beta}{D_T}\int_{\mathbb{R}}\frac{d\xi}{1+\xi^2}.$$

Consequently, there exists $\sigma \in (0,\tau)$ such that

$$\int_{\mathbb{R}} |\hat f(\sigma,\xi)|^2\,|\xi|^\alpha(1+\xi^2)^{-1}\,d\xi < \infty.$$

Since $(1+\xi^2)^{-1+\alpha/2} \le 2^{\alpha/2}\,(1+\xi^2)^{-1}(1+|\xi|^\alpha)$, we deduce that

$$\int_{\mathbb{R}} |\hat f(\sigma,\xi)|^2\,(1+\xi^2)^{-1+\alpha/2}\,d\xi < \infty.$$

Taking $\beta = -1+\alpha/2$ in (81), integrating and letting $R\to+\infty$, we now obtain that, for every $s \in (\sigma, T)$,

$$\begin{aligned}\int_{\mathbb{R}} |\hat f(s,\xi)|^2\,(1+\xi^2)^{-1+\alpha/2}\,d\xi &\le e^{F_\beta(s-\sigma)}\int_{\mathbb{R}} |\hat f(\sigma,\xi)|^2\,(1+\xi^2)^{-1+\alpha/2}\,d\xi \\ \int_\sigma^T\int_{\mathbb{R}} |\hat f(t,\xi)|^2\,|\xi|^\alpha(1+\xi^2)^{-1+\alpha/2}\,d\xi\,dt &\le \frac{1+e^{F_\beta(T-\sigma)}}{D_T}\int_{\mathbb{R}} |\hat f(\sigma,\xi)|^2\,(1+\xi^2)^{-1+\alpha/2}\,d\xi.\end{aligned}$$

We may thus proceed as previously. By induction, we conclude that, for every $\beta \ge -1$, (72) holds, which completes the proof of Theorem 1.

4 Uniqueness and large time behaviour

This section is devoted to the proof of Theorem 2 and Theorem 4. To this aim, we consider two solutions f and g to (1), (3) such that (5) holds. Let $t \geq 0$. The Fourier transforms $\hat{f}$ and $\hat{g}$ both satisfy (73). As in [35], we set

$$u(t,\xi) = \hat{f}(t,\xi) - \hat{g}(t,\xi) + i(\zeta_f(t) - \zeta_g(t))\,\phi(\xi), \tag{82}$$

where ϕ denotes a smooth bounded odd function that satisfies

$$\phi(\xi) = \xi \quad \text{for } |\xi| \leq 1 \quad \text{and} \quad \phi(\xi) = \begin{cases} -2 & \text{if } \xi \leq -3 \\ 2 & \text{if } \xi \geq 3. \end{cases}$$

Such a function ϕ has been chosen so that u satisfies

$$u(t,0) = 0, \qquad \partial_\xi u(t,0) = 0 \qquad \text{and} \qquad |\partial^2_{\xi,\xi} u(t,\xi)| \leq 2\,(1 + \sup_{[-3,3]} |\phi''|),$$

for every $\xi \in \mathbb{R}$, which implies that the map $\xi \mapsto \frac{u(t,\xi)}{|\xi|^2}$ is well-defined and bounded on $\mathbb{R}$. Similar manipulations had already been done in [6]. Then, by (11) and (73), we have

$$\begin{aligned}\partial_t u(t,\xi) + iE\,\xi\,u(t,\xi) + E\,\zeta_g(t)\,\xi\,\partial_\xi u(t,\xi) &= (\zeta_f(t) - \zeta_g(t))\;(\xi\,R_1(t,\xi) + R_2(t,\xi)) \\ &\quad - iK\,(\zeta_f(t) - \zeta_g(t))\,\phi(\xi) + \hat{Q}(\hat{f},\hat{f})(t,\xi) - \hat{Q}(\hat{g},\hat{g})(t,\xi),\end{aligned}$$

where

$$\begin{aligned} R_1(t,\xi) &= -E\,\left(\partial_\xi \hat{f}(t,\xi) + i\zeta_f(t)\,\phi'(\xi) + \phi(\xi)\right), \\ R_2(t,\xi) &= iE\,(\zeta_f(t) + \zeta_g(t))\,\left(\xi\,\phi'(\xi) - \phi(\xi)\right). \end{aligned}$$

We now prove the following proposition.

Proposition 9 *Consider $E > 0$ and a cross section b satisfying (2). Denote by f and g two weak solutions to (1) in the sense of Definition 1. There exists a constant $C > 0$, that only depends on E, ϕ and $\int_{-\pi}^{\pi} (\sin^2\theta + 1 - \cos\theta)\,b(\theta)\,d\theta$ such that the function u defined by (82) satisfies, for every $t \geq 0$, $\xi \in \mathbb{R}$,*

$$|u(t,\xi)| \leq |\xi|^2\,e^{-2EZ_g(t)} \left(\sup_{\xi\in\mathbb{R}} \frac{|u(0,\xi)|}{|\xi|^2} + C \int_0^t (\zeta_f(s) - \zeta_g(s))\,e^{2EZ_g(s)}\,ds \right),$$

where $Z_g(t) = \int_0^t \zeta_g(s)\,ds$.

Proof. The proof is inspired on the one hand from [6, 35] and on the other hand from [32]. For $n \in \mathbb{N}$, we set $b_n := \min\{b, n\}$. We then have $b = b_n + (b - b_n)$, which enables us to split the collision operator in two parts, one involving b_n where there is no more singularity in θ and one involving

$b-b_n$, which thus concerns small values of θ, more precisely, $|\theta| \le n^{-1/(1+\alpha)}$. Then,

$$\begin{aligned}
&\partial_t u(t,\xi) + \left(iE\,\xi + \|b_n\|_1 + \int_{-\pi}^{\pi} (1-\cos\theta)\,(b(\theta)-b_n(\theta))\,d\theta \right) u(t,\xi) + E\,\zeta_g(t)\,\xi\,\partial_\xi u(t,\xi) \\
&\quad = \int_{-\pi}^{\pi} \Big(u(t,\xi\cos\theta)\,\hat f(t,\xi\sin\theta) + \hat g(t,\xi\cos\theta)\,u(t,\xi\sin\theta) \Big)\, b_n(\theta)\,d\theta \\
&\quad + \int_{-\pi}^{\pi} \hat f(t,\xi\cos\theta)\,u(t,\xi\sin\theta)\,(b(\theta)-b_n(\theta))\,d\theta \\
&\quad + (\zeta_f(t)-\zeta_g(t))\,\left(\xi\, R_1(t,\xi) + R_2(t,\xi) + R_3(t,\xi)\right) + \; S_1(t,\xi) + S_2(t,\xi) + S_3(t,\xi),
\end{aligned}$$

where

$$\begin{aligned}
R_3(t,\xi) &= -i\int_{-\pi}^{\pi} \Big(\phi(\xi\cos\theta)\hat f(t,\xi\sin\theta) - \cos\theta\phi(\xi)\Big)\, b_n(\theta)\,d\theta, \\
S_1(t,\xi) &= \int_{-\pi}^{\pi} (\hat g(t,\xi\sin\theta) - 1 + i\xi\sin\theta\,\zeta_g(t))\,(\hat f(t,\xi\cos\theta) - \hat g(t,\xi\cos\theta))\,(b(\theta)-b_n(\theta))\,d\theta \\
&+ \; i\,(\zeta_f(t)-\zeta_g(t)) \int_{-\pi}^{\pi} \hat f(t,\xi\sin\theta)\,(\xi\sin\theta - \phi(\xi\sin\theta))\,(b(\theta)-b_n(\theta))\,d\theta, \\
S_2(t,\xi) &= \int_{-\pi}^{\pi} (\hat f(t,\xi\cos\theta) - \hat g(t,\xi\cos\theta) - \cos\theta(\hat f(t,\xi) - \hat g(t,\xi)))\,(b(\theta)-b_n(\theta))\,d\theta, \\
S_3(t,\xi) &= i\,\zeta_g(t) \int_{-\pi}^{\pi} \xi\sin\theta\,(\hat g(t,\xi\cos\theta) - \hat g(t,\xi))\,(b(\theta)-b_n(\theta))\,d\theta \\
&- \; i\,\zeta_f(t) \int_{-\pi}^{\pi} \xi\sin\theta\,(\hat f(t,\xi\cos\theta) - \hat f(t,\xi))\,(b(\theta)-b_n(\theta))\,d\theta.
\end{aligned}$$

We first show bounds for R_1, R_2, R_3, S_1, S_2 and S_3. The Taylor formula leads to

$$\begin{aligned}
R_1(t,\xi) &= -E\,\xi \left(\int_0^1 \Big(\partial^2_{\xi,\xi} \hat f(t,\lambda\xi) + i\zeta_f(t)\phi''(\lambda\xi) + \phi'(\lambda\xi) \Big)\, d\lambda \right), \\
R_2(t,\xi) &= iE\,(\zeta_f(t)+\zeta_g(t))\,\xi^2 \int_0^1 \lambda\,\phi''(\lambda\xi)\,d\lambda.
\end{aligned}$$

Consequently, we obtain that

$$|R_1(t,\xi)| \le E\,(1 + \|\phi''\|_\infty + \|\phi'\|_\infty)\,|\xi| \qquad \text{and} \qquad |R_2(t,\xi)| \le E\,\|\phi''\|_\infty\,|\xi|^2. \tag{83}$$

We now consider R_3. It reads

$$\begin{aligned}
R_3(t,\xi) &= -\frac{i}{2} \int_{-\pi}^{\pi} \phi(\xi\cos\theta)\,(\hat f(t,\xi\sin\theta) + \hat f(t,-\xi\sin\theta) - 2)\,b_n(\theta)\,d\theta \\
&- \; i\int_{-\pi}^{\pi} (\phi(\xi\cos\theta) - \cos\theta\,\phi(\xi))\,b_n(\theta)\,d\theta.
\end{aligned}$$

By the Taylor formula, we have

$$\begin{aligned}
\hat f(t,\xi\sin\theta) + \hat f(t,-\xi\sin\theta) - 2 &= \xi^2\,\sin^2\theta \int_{-1}^{1} (1-|\lambda|)\,\partial^2_{\xi,\xi}\hat f(t,\lambda\xi\sin\theta)\,d\lambda, \\
\phi(\xi\cos\theta) - \cos\theta\,\phi(\xi) &= -\xi^2\,\cos\theta\,(1-\cos\theta) \int_0^1 \lambda \int_0^1 \phi''(\lambda\xi(1+w(\cos\theta-1)))\,dw\,d\lambda.
\end{aligned}$$

Consequently, we deduce that

$$|R_3(t,\xi)| \le \frac{1}{2}\,(\|\phi\|_\infty + \|\phi''\|_\infty)\,|\xi|^2 \left(\int_{-\pi}^{\pi} (\sin^2\theta + 1 - \cos\theta)\, b_n(\theta)\, d\theta\right). \tag{84}$$

On the other hand, the Taylor formula ensures that

$$\begin{aligned} \hat g(t,\xi\sin\theta) - 1 + i\xi\sin\theta\zeta_g(t) &= \xi^2\sin^2\theta \int_0^1 (1-u)\,\partial^2_{\xi,\xi}\hat g(t,u\xi\sin\theta)\,du, \\ \xi\,\sin\theta - \phi(\xi\sin\theta) &= \xi^2\sin^2\theta \int_0^1 (u-1)\,\phi''(u\xi\sin\theta)\,du, \end{aligned}$$

and thus

$$|S_1(t,\xi)| \le (1 + \|\phi''\|_\infty)\,|\xi|^2 \int_{-\pi}^{\pi} \sin^2\theta\,(b(\theta) - b_n(\theta))\,d\theta. \tag{85}$$

Setting $j(t,\xi) = \hat f(t,\xi) - \hat g(t,\xi)$, we have $j(t,0) = 0$, $|\partial^2_{\xi,\xi} j(t,\xi)| \le 2$ and

$$j(t,\xi\cos\theta) - \cos\theta j(t,\xi) = -\xi^2\,\cos\theta\,(1-\cos\theta) \int_0^1 \lambda \int_0^1 \partial^2_{\xi,\xi} j(t,\lambda\xi(1 + w(\cos\theta - 1)))\,dw\,d\lambda.$$

Therefore, we obtain that

$$|S_2(t,\xi)| \le |\xi|^2 \int_{-\pi}^{\pi} (1-\cos\theta)\,(b(\theta) - b_n(\theta))\,d\theta. \tag{86}$$

It only remains to consider S_3. We have

$$|\hat g(t,\xi\cos\theta) - \hat g(t,\xi)| \le (1-\cos\theta)\,|\xi| \quad \text{and} \quad \left|\hat f(t,\xi\cos\theta) - \hat f(t,\xi)\right| \le (1-\cos\theta)\,|\xi|,$$

which imply that

$$|S_3(t,\xi)| \le 2\,|\xi|^2 \int_{-\pi}^{\pi} (1-\cos\theta)\,(b(\theta) - b_n(\theta))\,d\theta. \tag{87}$$

We now set $Z_g(t) = \int_0^t \zeta_g(s)\,ds$, and, for any two-variables function η, we denote by $\eta^\#$ the function defined by $\eta^\#(t,\xi) = \eta(t, e^{EZ_g(t)}\,\xi)$. We next put

$$w(t,\xi) = u^\#(t,\xi)\,e^{t(\|b_n\|_1 + \varepsilon_n)}, \qquad \text{where} \quad \varepsilon_n = \int_{-\pi}^{\pi} (1-\cos\theta)\,(b(\theta) - b_n(\theta))\,d\theta. \tag{88}$$

With these notations, it is easily checked that w satisfies

$$\begin{aligned} \partial_t w(t,\xi) + iEe^{EZ_g(t)}\xi w(t,\xi) &= \int_{-\pi}^{\pi} \hat f^\#(t,\xi\cos\theta)\,w(t,\xi\sin\theta)\,(b(\theta) - b_n(\theta))\,d\theta \\ &+ \int_{-\pi}^{\pi} \Big(w(t,\xi\cos\theta)\,\hat f^\#(t,\xi\sin\theta) + \hat g^\#(t,\xi\cos\theta)\,w(t,\xi\sin\theta)\Big)\,b_n(\theta)\,d\theta \\ &+ e^{t(\|b_n\|_1 + \varepsilon_n)}\,(\zeta_f(t) - \zeta_g(t))\,\Big(e^{EZ_g(t)}\,\xi\,R_1^\#(t,\xi) + R_2^\#(t,\xi) + R_3^\#(t,\xi)\Big) \\ &+ e^{t(\|b_n\|_1 + \varepsilon_n)}\,(S_1^\#(t,\xi) + S_2^\#(t,\xi) + S_3^\#(t,\xi)). \end{aligned}$$

Consequently, setting $H(t,\xi) = \exp\left(iE\xi \int_0^t e^{EZ_g(s)}\,ds\right)$, we obtain

$$w(t,\xi)H(t,\xi) = w(0,\xi) + \int_0^t H(s,\xi) \int_{-\pi}^{\pi} \hat{f}^{\#}(s,\xi\cos\theta)\, w(s,\xi\sin\theta)\,(b(\theta)-b_n(\theta))\,d\theta\,ds$$
$$+ \int_0^t H(s,\xi) \int_{-\pi}^{\pi} \Big(w(s,\xi\cos\theta)\,\hat{f}^{\#}(s,\xi\sin\theta) + \hat{g}^{\#}(s,\xi\cos\theta)\,w(s,\xi\sin\theta)\Big)\, b_n(\theta)\,d\theta\,ds$$
$$+ \int_0^t H(s,\xi)\, e^{s(\|b_n\|_1+\varepsilon_n)}\,(\zeta_f(s)-\zeta_g(s))\,\Big(e^{EZ_g(s)}\,\xi\, R_1^{\#}(s,\xi) + R_2^{\#}(s,\xi) + R_3^{\#}(s,\xi)\Big)\,ds$$
$$+ \int_0^t H(s,\xi)\, e^{s(\|b_n\|_1+\varepsilon_n)}\,(S_1^{\#}(s,\xi) + S_2^{\#}(s,\xi) + S_3^{\#}(s,\xi))\,ds.$$

It therefore follows that

$$\begin{aligned}
|w(t,\xi)| \;\le\;& |w(0,\xi)| + \int_0^t \int_{-\pi}^{\pi} |w(s,\xi\sin\theta)|\,(b(\theta)-b_n(\theta))\,d\theta\,ds \\
&+ \int_0^t \int_{-\pi}^{\pi} (|w(s,\xi\cos\theta)| + |w(s,\xi\sin\theta)|)\,b_n(\theta)\,d\theta\,ds \\
&+ \int_0^t e^{s(\|b_n\|_1+\varepsilon_n)}\,(\zeta_f(s)-\zeta_g(s))\,\Big(e^{EZ_g(s)}\,|\xi\, R_1^{\#}(s,\xi)| + |R_2^{\#}(s,\xi)| + |R_3^{\#}(s,\xi)|\Big)\,ds \\
&+ \int_0^t e^{s(\|b_n\|_1+\varepsilon_n)}\,(|S_1^{\#}(s,\xi)| + |S_2^{\#}(s,\xi)| + |S_3^{\#}(s,\xi)|)\,ds.
\end{aligned}$$

We now deduce from (83), (84), (85), (86) and (87) that

$$\frac{|w(t,\xi)|}{|\xi|^2} \le \frac{|w(0,\xi)|}{|\xi|^2} + \int_0^t \int_{-\pi}^{\pi} \frac{|w(s,\xi\sin\theta)|}{|\xi\sin\theta|^2}\,|\sin\theta|^2\,(b(\theta)-b_n(\theta))\,d\theta\,ds$$
$$+ \int_0^t \int_{-\pi}^{\pi} \left(\frac{|w(s,\xi\cos\theta)|}{|\xi\cos\theta|^2}\,|\cos\theta|^2 + \frac{|w(s,\xi\sin\theta)|}{|\xi\sin\theta|^2}\,|\sin\theta|^2 \right) b_n(\theta)\,d\theta\,ds$$
$$+ C\left(1 + \int_{-\pi}^{\pi} (\sin^2\theta + 1 - \cos\theta)\,b_n(\theta)\,d\theta\right) \int_0^t (\zeta_f(s)-\zeta_g(s))\,e^{s(\|b_n\|_1+\varepsilon_n)+2EZ_g(s)}\,ds$$
$$+ C\left(\int_{-\pi}^{\pi} (\sin^2\theta + 1 - \cos\theta)\,(b(\theta)-b_n(\theta))\,d\theta\right) \int_0^t e^{s(\|b_n\|_1+\varepsilon_n)+2EZ_g(s)}\,ds,$$

where C denotes a constant that only depends on E and ϕ. Setting

$$\alpha_n = \int_{-\pi}^{\pi} |\sin\theta|^2\,(b(\theta)-b_n(\theta))\,d\theta + \int_{-\pi}^{\pi} b_n(\theta)\,d\theta,$$

we thus have, for $X > 0$,

$$\sup_{|\xi|\le X} \frac{|w(t,\xi)|}{|\xi|^2} \le \sup_{|\xi|\le X} \frac{|w(0,\xi)|}{|\xi|^2} + \alpha_n \int_0^t \sup_{|\xi|\le X} \frac{|w(s,\xi)|}{|\xi|^2}\,ds$$
$$+ C\left(1 + \int_{-\pi}^{\pi} (\sin^2\theta + 1 - \cos\theta)\,b_n(\theta)\,d\theta\right) \int_0^t (\zeta_f(s)-\zeta_g(s))\,e^{s(\|b_n\|_1+\varepsilon_n)+2EZ_g(s)}\,ds$$
$$+ C\left(\int_{-\pi}^{\pi} (\sin^2\theta + 1 - \cos\theta)\,(b(\theta)-b_n(\theta))\,d\theta\right) \int_0^t e^{s(\|b_n\|_1+\varepsilon_n)+2EZ_g(s)}\,ds.$$

The Gronwall lemma then ensures that

$$\sup_{|\xi|\le X}\frac{|w(t,\xi)|}{|\xi|^2}\le e^{\alpha_n t}\sup_{|\xi|\le X}\frac{|w(0,\xi)|}{|\xi|^2}$$
$$+C\left(1+\int_{-\pi}^{\pi}(\sin^2\theta+1-\cos\theta)\,b_n(\theta)\,d\theta\right)\int_0^t(\zeta_f(s)-\zeta_g(s))\,e^{\alpha_n(t-s)}\,e^{s(\|b_n\|_1+\varepsilon_n)+2EZ_g(s)}\,ds$$
$$+C\left(\int_{-\pi}^{\pi}(\sin^2\theta+1-\cos\theta)\,(b(\theta)-b_n(\theta))\,d\theta\right)\int_0^t e^{\alpha_n(t-s)}\,e^{s(\|b_n\|_1+\varepsilon_n)+2EZ_g(s)}\,ds.$$

Recalling that w is defined by (88), we now obtain that

$$\sup_{|\xi|\le X}\frac{|u(t,e^{EZ_g(t)}\xi)|}{|\xi|^2}\le e^{(\alpha_n-\|b_n\|_1-\varepsilon_n)t}\sup_{|\xi|\le X}\frac{|u(0,\xi)|}{|\xi|^2}$$
$$+C\left(1+\int_{-\pi}^{\pi}(\sin^2\theta+1-\cos\theta)\,b_n(\theta)\,d\theta\right)\int_0^t(\zeta_f(s)-\zeta_g(s))\,e^{(\alpha_n-\|b_n\|_1-\varepsilon_n)(t-s)}\,e^{2EZ_g(s)}\,ds$$
$$+C\left(\int_{-\pi}^{\pi}(\sin^2\theta+1-\cos\theta)\,(b(\theta)-b_n(\theta))\,d\theta\right)\int_0^t e^{(\alpha_n-\|b_n\|_1-\varepsilon_n)(t-s)}\,e^{2EZ_g(s)}\,ds.$$

But, $\lim_{n\to+\infty}\varepsilon_n=0$ and $\lim_{n\to+\infty}(\alpha_n-\|b_n\|_1)=0$. Hence, passing to the limit $n\to+\infty$ leads to

$$\begin{aligned}\sup_{|\xi|\le X}\frac{|u(t,e^{EZ_g(t)}\xi)|}{|\xi|^2}\quad&\le\quad\sup_{|\xi|\le X}\frac{|u(0,\xi)|}{|\xi|^2}\\&+\quad C\left(1+\int_{-\pi}^{\pi}(\sin^2\theta+1-\cos\theta)\,b(\theta)\,d\theta\right)\int_0^t(\zeta_f(s)-\zeta_g(s))\,e^{2EZ_g(s)}\,ds.\end{aligned}$$

Consequently, for $X>0$, we have

$$\sup_{|\xi|\le X}\frac{|u(t,e^{EZ_g(t)}\xi)|}{|\xi|^2}\le\left(\sup_{\xi\in\mathbb{R}}\frac{|u(0,\xi)|}{|\xi|^2}+C\int_0^t(\zeta_f(s)-\zeta_g(s))\,e^{2EZ_g(s)}\,ds\right),$$

where the constant C only depends on E, ϕ and $\int_{-\pi}^{\pi}(\sin^2\theta+1-\cos\theta)\,b(\theta)\,d\theta$. The right-hand side of the previous inequality being independent of X, we obtain that the desired result holds for every $\xi\in\mathbb{R}$. □

Proof of Theorem 2. Let f_{in} and g_{in} be two functions satisfying (4) and (8). Denote by f and g two weak solutions to (1) with initial conditions respectively f_{in} and g_{in}. Then, $\zeta_f\equiv\zeta_g$. We thus deduce from Proposition 9 that

$$|\hat f(t,\xi)-\hat g(t,\xi)|\le|\xi|^2\,e^{-2EZ_g(t)}\sup_{\xi\in\mathbb{R}}\frac{|\hat f_{in}(\xi)-\hat g_{in}(\xi)|}{|\xi|^2},$$

for every $\xi\in\mathbb{R}$ and $t\ge0$, which completes the proof of Theorem 2. □

Proof of Theorem 4. Taking $g = f_{stat}$ in Proposition 9 and recalling that u is defined by (82), we get

$$
\begin{aligned}
|\hat{f}(t,\xi) - \hat{f}_{stat}(\xi)| &\leq |u(t,\xi)| + |\zeta_f(t) - \zeta_g(t)|\, |\phi(\xi)| \\
&\leq |\xi|^2\, e^{-2EZ_g(t)} \left(\sup_{\xi\in\mathbb{R}} \frac{|u(0,\xi)|}{|\xi|^2} + C \int_0^t (\zeta_f(s) - \zeta_g(s))\, e^{2EZ_g(s)}\, ds \right) \\
&+ C\, |\xi|\, |\zeta_f(t) - \zeta_g(t)|,
\end{aligned}
$$

for every $t \geq 0$ and $\xi \in \mathbb{R}$. Besides, we have $\zeta_g \equiv \zeta_+$, where ζ_+ is given by (14) and we infer from (13) that

$$
|\zeta_f(s) - \zeta_+| \leq C\, e^{-s\sqrt{K^2+4E^2}},
$$

for every $s \geq 0$. Consequently,

$$
\frac{|\hat{f}(t,\xi) - \hat{f}_{stat}(\xi)|}{|\xi|} \leq |\xi|\, e^{-(\sqrt{K^2+4E^2}-K)t} \left(\sup_{\xi\in\mathbb{R}} \frac{|u(0,\xi)|}{|\xi|^2} + C \int_0^t e^{-Ks}\, ds \right) + C\, e^{-t\sqrt{K^2+4E^2}},
$$

for every $t \geq 0$ and $\xi \in \mathbb{R}$. Thus, we have, for $t \geq 0$ and $R > 0$,

$$
\sup_{|\xi|\leq R} \frac{|\hat{f}(t,\xi) - \hat{f}_{stat}(\xi)|}{|\xi|} \leq C\, R\, e^{-(\sqrt{K^2+4E^2}-K)t} + C\, e^{-t\sqrt{K^2+4E^2}}.
$$

On the other hand, since f and f_{stat} both have mass 1, we obtain that

$$
\sup_{|\xi|\geq R} \frac{|\hat{f}(t,\xi) - \hat{f}_{stat}(\xi)|}{|\xi|} \leq \frac{2}{R}.
$$

Combining the two previous inequalities, we deduce that

$$
d_1(f(t,.), f_{stat}) \leq C\, R\, e^{-(\sqrt{K^2+4E^2}-K)t} + C\, e^{-t\sqrt{K^2+4E^2}} + \frac{2}{R}.
$$

Then, taking

$$
R = \sqrt{\frac{2}{C\, e^{-(\sqrt{K^2+4E^2}-K)t}}},
$$

completes the proof of Theorem 4. □

A Auxiliary results on convex functions

Let $\Phi \in \mathcal{C}^1([0,+\infty))$ be a nonnegative convex function such that $\Phi(0) = 0$, $\Phi'(0) = 0$, Φ' is concave.

Lemma 10 *For $r \in [0,+\infty)$ and $\lambda \in [0,+\infty)$,*

$$
\Phi(\lambda r) \leq \max\{1, \lambda^2\}\, \Phi(r).
$$

Proof. Let $r \geq 0$. If $\lambda \in [0,1]$ then the monotonicity of Φ ensures that

$$\Phi(\lambda r) \leq \Phi(r) \leq \max\{1, \lambda^2\}\, \Phi(r).$$

Otherwise, we deduce from the concavity and the nonnegativity of Φ' that

$$\Phi'(s) = \Phi'\left(\frac{\lambda s}{\lambda} + \left(1 - \frac{1}{\lambda}\right) 0\right) \geq \frac{1}{\lambda}\, \Phi'(\lambda s),$$

for $s \in [0,r]$. Integrating this inequality over $(0,r)$ then leads to

$$\Phi(r) \geq \frac{1}{\lambda^2}\, \Phi(\lambda r),$$

which completes the proof of Lemma 10. $\square$

Lemma 11 *The mapping* $r \mapsto \frac{\Phi(r)}{r}$ *is concave.*

Proof. Consider $r \geq 0$, $s \geq 0$ and $\lambda \in [0,1]$. Since Φ' is concave, we have

$$\begin{aligned} \frac{\Phi(\lambda r + (1-\lambda)s)}{\lambda r + (1-\lambda)s} &= \int_0^1 \Phi'(z(\lambda r + (1-\lambda)s))\, dz \\ &\geq \int_0^1 (\lambda \Phi'(zr) + (1-\lambda)\Phi'(zs))\, dz \\ &\geq \lambda\, \frac{\Phi(r)}{r} + (1-\lambda)\, \frac{\Phi(s)}{s}. \end{aligned}$$

$\square$

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