Stationary states for the non-cutoff Kac equation with a Gaussian thermostat

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Abstract

We study the stationary states of a Kac equation with a Gaussian thermostat in the case of a non-cutoff cross section. We investigate the existence, smoothness and uniqueness of the stationary states. The theoretical results are illustrated by some numerical simulations.

1 Introduction

We consider the non-cutoff Kac equation with a thermostated force field

$$\partial_t f + E\, \partial_v((1-\zeta(t)v)f) = Q(f,f), \tag{1}$$

where $\zeta(t) = \int_{\mathbb{R}} v f(t,v)\, dv$ and

$$Q(f,f)(t,v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} (f(t,v')f(t,v'_*) - f(t,v)f(t,v_*))\, b(\theta)\, d\theta\, dv_*, \tag{2}$$

with

$$v' = v\cos\theta - v_*\sin\theta, \qquad v'_* = v\sin\theta + v_*\cos\theta,$$

and

$$b(\theta) = |\theta|^{-1-\alpha}, \qquad \theta \in (-\pi,\pi), \qquad \alpha \in (0,2). \tag{3}$$

The right-hand side is the collision term in Mark Kac's one-dimensional caricature of the Boltzmann equation, and the left-hand side comes from a thermostated force field, which we describe next.

Kac's original equation is derived from the evolution of a stochastic N-particle system, in which the velocity of an individual particle is one-dimensional, and the positions are neglected (see [5]). The system is energy conserving, and therefore the phase space is $\mathbb{S}^{N-1}$. In the original model collisions are modelled by random rotations of randomly chosen pairs of velocities: $(v_j, v_k) \mapsto (v_j\cos\theta - v_k\sin\theta, v_j\sin\theta + v_k\cos\theta)$, and originally θ was chosen uniformly in $[-\pi,\pi]$ (corresponding to $b(\theta) = 1/(2\pi)$ in (2)), and the intervals between collisions were taken to be exponentially distributed, with a parameter proportional to N.

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If the particles are also accelerated by a force field, $dv_j/dt = E$, the system is no longer conservative, but energy conservation can be recovered by projecting the complete force field onto the tangent space of $\mathbb{S}^{N-1}$. This construction is known as a Gaussian isokinetic thermostat, and has been applied in many fields of statistical physics and molecular dynamics, as a model for non equilibrium steady states (see [2, 6, 7, 8] and the references therein).

With the thermostated field and the collisions, a phase space density will evolve according to the following master equation, in which it is assumed that $\sum_{i=1}^N v_i^2 = N$:

$$\partial_t \psi_N(t, \mathbf{V}) + E \sum_{j=1}^{N} \frac{\partial}{\partial v_j} \left((1 - J v_j)\, \psi_N(t, \mathbf{V}) \right) = \binom{N}{2}^{-1} \sum_{1 \le i < j \le N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\psi_N\big(t, R_{ij}(\theta)\mathbf{V}\big) - \psi_N(t, \mathbf{V}) \right) d\theta, \quad (4)$$

where $\mathbf{V} = (v_1, ..., v_N)$, $NJ = \sum v_j$ and $R_{ij}(\theta)\mathbf{V} = (v_1, \ldots, v_i', \ldots, v_j', \ldots, v_N)$.

Equation (1), with $b(\theta) = 1/(2\pi)$ is obtained by computing the one particle marginals of the solution ψ_N to (4), and then letting N go to infinity. In that setting, (1) was considered first in [9], and the principal result is that the equation possesses a stationary solution, and that depending on the field strength E, the stationary solution may be either continuous, or have a power-like singularity. The reason is that there is a competition between the force field, which tries to concentrate the distribution function on a Dirac mass at $v = 1$, and the collision term which drives the distribution function towards a centered Maxwellian. Details of the derivation of (1) and some generalizations may be found in [10], which also deals with the time dependent problem.

A natural generalization of this is to replace the distribution of rotation angles by a density $b(\theta)$, and Desvillettes [3] introduced a model corresponding to the Boltzmann equation for non-cutoff molecules, in which $b(\theta) \sim |\theta|^{-1-\alpha}$ with $0 < \alpha < 2$. In this case $b(\theta)$ is not integrable, and the collision frequency is infinite. However, for any $0 < \theta_1 < \theta_2$, $\int_{\theta_1 \le |\theta| \le \theta_2} b(\theta)\, d\theta$ is finite, and corresponds to the expected frequency of jumps with θ in the given interval. In this way, the collisions still form a Poisson process. Desvillettes, who was the first to consider this (with no force field), proved that the collision operator is smoothing in this case, and that the solutions to the time dependent problem immediately become smooth, much like in the heat equation. Also for the Boltzmann equation, the non-cutoff collision operator has a smoothing effect (see [1, 4]).

In this paper we consider the stationary case of (1):

$$E\, \frac{d}{dv}((1 - \zeta v) f(v)) = Q(f, f)(v), \qquad v \in \mathbb{R}. \quad (5)$$

We are interested in the question as to whether the regularizing effect of the non-cutoff collision operator is enough to prevent also a very strong force field to yield a singularity of the stationary solution. We show that this is the case, and in fact, it is not a surprising result. While with the cutoff collision operator, there are two distinct time scales that can be directly compared with each other: the mean time between collisions, and a time scale related to the acceleration of particles, only the time scale related to the acceleration remains in the non-cutoff case.

We now give a relevant definition of solutions in the non-cutoff situation, and state the main result of the paper.

Definition 1 *Assume that b satisfies (3). A function* $f \in L^1_2(\mathbb{R})$ *is said to be a weak solution to (5), (2) if it satisfies*

$$\int_{\mathbb{R}} (\zeta\, v - 1)\, f(v)\, \psi'(v)\, dv = \frac{1}{E} \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b(\theta)\, d\theta\, f(v)\, f(v_*)\, dv\, dv_*, \tag{6}$$

for every $\psi \in \mathcal{C}^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$.

Our main result is the following.

Theorem 1 *Assume that b satisfies (3). For all field strengths* $E > 0$, *there exists a unique weak solution f to (5), (2) such that moments of any order of f are finite and*

$$\int_{\mathbb{R}} f(v)\, dv = 1. \tag{7}$$

Moreover, $f \in \mathcal{C}^\infty(\mathbb{R})$.

The paper is organised as follows. First, in Section 2, we show that a solution to equation (5), (2) exists if the cross-section b is supposed to be Lipschitz continuous on $[-\pi, \pi]$. The proof is to a large extent an adaptation to the present case of the techniques used in [9]. As with the Boltzmann equation for Maxwellian molecules, much is simplified by the possibility to compute moments exactly.

We then deduce in Section 3.1 the existence of a solution to (5), (2) for a cross-section b satisfying (3). The smoothness of such a solution is investigated in Section 3.2 by the use of Fourier transform techniques, much like in [3]. Section 3.3 is then devoted to the proof of the uniqueness part of Theorem 1.

We also illustrate, in Section 4, the theoretical results by some numerical simulations, obtained by solving the equation satisfied by the Fourier transform.

2 Cutoff case

We consider here the stationary equation (5), (2) when $b : [-\pi, \pi] \to \mathbb{R}$ is an even and Lipschitz continuous function. Without loss of generality, one can assume that $\int_{\mathbb{R}} f(v)\, dv = 1$. Then, the collision operator reads

$$Q(f, f)(v) = Q^+(f, f)(v) - \|b\|_{L^1} f(v), \qquad v \in \mathbb{R},$$

where

$$Q^+(f, f)(v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} f(v')\, f(v'_*)\, b(\theta)\, d\theta\, dv_*. \tag{8}$$

Since

$$\int_{\mathbb{R}} Q^+(f, f)(v)\, v\, dv = \zeta \int_{-\pi}^{\pi} \cos\theta\, b(\theta)\, d\theta,$$

we obtain, by multiplying (5) by v and integrating, that ζ satisfies $\zeta^2 + (K/E)\,\zeta - 1 = 0$, where $K := \int_{-\pi}^{\pi} (1 - \cos\theta)\, b(\theta)\, d\theta$. We then deduce that

$$\zeta = \frac{\sqrt{K^2 + 4E^2} - K}{2E}$$

is the only root that allows $\int_{\mathbb{R}} v^2 f(v)\, dv \le 1$. We set $\kappa = 1/\zeta$ and

$$\gamma = \frac{\|b\|_{L^1}}{E\,\zeta} - 1.$$

Dividing (5) by $(v-\kappa)|v-\kappa|^\gamma$, we obtain, for $v \neq \kappa$,

$$\frac{d}{dv}\left(\frac{1}{|v-\kappa|^\gamma}\, f(v)\right) = -\frac{\gamma+1}{\|b\|_{L^1}}\,\frac{1}{(v-\kappa)|v-\kappa|^\gamma}\, Q^+(f,f)(v).$$

Then, any solution to (5), (2) with $\int_{\mathbb{R}} f(v)\, dv = 1$ and $\int_{\mathbb{R}} v^2\, f(v)\, dv = 1$ satisfies

$$f(v) = \mathcal{A}(f)(v), \qquad v \in \mathbb{R}\backslash\{\kappa\}, \tag{9}$$

where

$$\mathcal{A}(f)(v) = \frac{\gamma+1}{\|b\|_{L^1}}\, |v-\kappa|^\gamma \begin{cases} \displaystyle\int_{-\infty}^{v} \frac{1}{|w-\kappa|^{\gamma+1}}\, Q^+(f,f)(w)\, dw & (v<\kappa) \\ \displaystyle\int_{v}^{\infty} \frac{1}{|w-\kappa|^{\gamma+1}}\, Q^+(f,f)(w)\, dw & (v>\kappa) \end{cases}$$

Theorem 2 *Let $b : [-\pi,\pi] \to \mathbb{R}$ be an even and lipschitz continuous function. For all field strengths $E > 0$, there exists a solution f to (9) such that $f \in \mathcal{C}(\mathbb{R}\backslash\{\kappa\})$, moments of any order of f are finite,*

$$\int_{\mathbb{R}} f(v)\, dv = 1, \qquad \int_{\mathbb{R}} f(v)\, v\, dv = \zeta \qquad \textit{and} \qquad \int_{\mathbb{R}} v^2\, f(v)\, dv = 1.$$

Moreover, for $\gamma > 0$, $f \in \mathcal{C}(\mathbb{R})$ and $Q^+(f,f) \in \mathcal{C}(\mathbb{R})$.

The proof of Theorem 2 is similar to that of [9, Theorem 1]. A solution to (9) is obtained by passing to the limit in the sequence generated by the iteration

$$f_{n+1} = \mathcal{A}(f_n). \tag{10}$$

Therefore, some estimates on $\mathcal{A}$ are needed. We first define

$$\Lambda(\psi)(w) = \frac{\gamma+1}{|w-\kappa|^{\gamma+1}} \int_\kappa^w (v-\kappa)\, |v-\kappa|^{\gamma-1}\, \psi(v)\, dv, \qquad w \in \mathbb{R},$$

and

$$\overline{\Lambda}(\psi)(v,v_*) = \int_{-\pi}^{\pi} \Lambda(\psi)(v')\, \frac{b(\theta)}{\|b\|_{L^1}}\, d\theta, \qquad (v,v_*) \in \mathbb{R}^2.$$

For $\psi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned} \int_{\mathbb{R}} \psi(v)\, \mathcal{A}(f)(v)\, dv &= \frac{\gamma+1}{\|b\|_{L^1}} \int_{-\infty}^{\kappa} \int_{-\infty}^{v} \frac{|v-\kappa|^\gamma}{|w-\kappa|^{\gamma+1}}\, Q^+(f,f)(w)\, \psi(v)\, dw\, dv \\ &\quad + \frac{\gamma+1}{\|b\|_{L^1}} \int_{\kappa}^{\infty} \int_{v}^{\infty} \frac{|v-\kappa|^\gamma}{|w-\kappa|^{\gamma+1}}\, Q^+(f,f)(w)\, \psi(v)\, dw\, dv. \end{aligned}$$

Thus, changing the order of integration, we obtain

$$\int_{\mathbb{R}} \psi(v)\, \mathcal{A}(f)(v)\, dv = \int_{\mathbb{R}} \Lambda(\psi)(w)\, Q^+(f,f)(w)\, \frac{dw}{\|b\|_{L^1}}.$$

By (8), we deduce that

$$\int_{\mathbb{R}} \psi(v)\, \mathcal{A}(f)(v)\, dv = \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v, v_*)\, f(v)\, f(v_*)\, dv\, dv_*.$$

Since, for any bounded measure μ, $v \mapsto \int_{\mathbb{R}} \overline{\Lambda}(\psi)(v, v_*)\, \mu(dv_*)$ is a bounded and continuous function, the mapping $f \mapsto \mathcal{A}(f)$ extends to a mapping $\mu \mapsto \mathcal{A}(\mu)$ on the space of bounded measures. A first step in the proof of Theorem 2 consists in a computation of the moments of $\mathcal{A}(\mu)$.

Lemma 3 *Let $m \in \mathbb{N}_*$. Then, for any measure μ satisfying $\int_{\mathbb{R}} \mu(dv) = 1$, we have*

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{A}(\mu)(dv) &= 1, \\
\int_{\mathbb{R}} v\, \mathcal{A}(\mu)(dv) &= \int_{\mathbb{R}} v\, \mu(dv) + \frac{1}{\gamma+2}\left(1 - \frac{K\,(\gamma+3)}{\|b\|_{L^1}}\right)\left(\zeta - \int_{\mathbb{R}} v\, \mu(dv)\right), \\
\int_{\mathbb{R}} v^2 \mathcal{A}(\mu)(dv) &= \int_{\mathbb{R}} v^2 \mu(dv) + \frac{2\,\kappa}{\gamma+2}\left(1 - \frac{K}{\|b\|_{L^1}}\right)\left(\zeta - \int_{\mathbb{R}} v\, \mu(dv)\right) \\
&\quad + \frac{2}{\gamma+3}\left[\left(1 - \int_{\mathbb{R}} v^2 \mu(dv)\right) - 2\,\kappa\left(1 - \frac{K}{\|b\|_{L^1}}\right)\left(\zeta - \int_{\mathbb{R}} v\, \mu(dv)\right)\right], \\
\int_{\mathbb{R}} v^m \mathcal{A}(\mu)(dv) &= \left(1 - \frac{m}{m+\gamma+1}\right)\int_{-\pi}^{\pi} (\cos^m\theta + \sin^m\theta)\, \frac{b(\theta)}{\|b\|_{L^1}}\, d\theta \int_{\mathbb{R}} v^m \mu(dv) \\
&\quad + M_{m-1},
\end{aligned} \tag{11}$$

where M_{m-1} only depends on moments of order $\leq m-1$.

Proof. The proof of this lemma is similar to that of [9, Lemma 2]. Let $m \in \mathbb{N}_*$. For $\psi(v) = v^m$, we have

$$\begin{aligned}
\Lambda(\psi)(v) &= v^m - \frac{m}{|v-\kappa|^{\gamma+1}} \int_{\kappa}^{v} |w-\kappa|^{\gamma+1} w^{m-1}\, dw \\
&= v^m - \frac{m}{\gamma+m+1}(v-\kappa)^m + \frac{m}{|v-\kappa|^{\gamma+1}} \int_{\kappa}^{v} |w-\kappa|^{\gamma+1} P_{m-2}(w)\, dw
\end{aligned}$$

where $P_{m-2}(w) = (w-\kappa)^{m-1} - w^{m-1}$ is a polynomial of degree $m-2$. For $m = 1$ and $m = 2$, we obtain, respectively,

$$\begin{aligned}
\Lambda(\psi)(v) &= \left(1 - \frac{1}{\gamma+2}\right) v + \frac{\kappa}{\gamma+2}, \\
\Lambda(\psi)(v) &= \left(1 - \frac{2}{\gamma+3}\right) v^2 + \left(\frac{-2\kappa}{\gamma+2} + \frac{4\kappa}{\gamma+3}\right) v + \left(\frac{2\kappa^2}{\gamma+2} - \frac{2\kappa^2}{\gamma+3}\right).
\end{aligned}$$

Then, Lemma 3 follows easily. □

Remark 4 *Lemma 3 ensures that the iteration (10) gives a tight set of unit measures. By weak compactness, there exists a measure μ such that, up to an extraction, $f_n \rightharpoonup \mu$.*

We now investigate the singularity of $\mathcal{A}(\mu)$ at $v = \kappa$. In the rest of this paper, C denotes a positive constant that may vary from one step to the other.

Lemma 5 *Let us assume that $f \in L^1(\mathbb{R})$ satisfies $\int_{\mathbb{R}} v^2\, f(v)\, dv < \infty$ and $f(v) \le C'|v-\kappa|^{-1}$, for some constant $C' > 0$. Then there is a constant $C > 0$ such that*

$$Q^+(f,f)(v) \le C(1 + (\log|v-\kappa|)^2),$$

in a neighbourhood of $v = \kappa$. The constant C depends only on κ and $\|b\|_{L^\infty}$.

Proof. The proof is carried out by rather straightforward estimates of the integrals in Q^+, similar to that of [9, Lemma 3]. We only need to note that, since $b \in L^\infty(-\pi,\pi)$, we have

$$Q^+(f,f)(v) \le \|b\|_{L^\infty} \int_{\mathbb{R}} \int_{-\pi}^{\pi} f(v')\, f(v'_*)\, d\theta\, dv_*.$$

□

Lemma 6 *Let $f \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} v^2 f(v)\, dv < \infty$ and $f(v) \le C(1 + (\log|v-\kappa|)^2)$ for some constant $C > 0$. Then, $Q^+(f,f)$ is bounded in a neighbourhood of $v = \kappa$.*

Moreover, if $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then $Q^+(f,f)$ is Hölder-continuous with exponent $1/2$ on each ball $B(0,R)$ with center 0 and radius R. The constant of Hölder-continuity depends only on R, $\|f\|_{L^\infty}$, $\|b\|_{L^\infty}$ and the lipschitz constant of b.

Proof. The L^∞-bound of the cross-section b enables us to proceed as in [9, Lemma 4] to prove the boudedness of $Q^+(f,f)$ in a neighbourhood of $v = \kappa$. However, the proof of the Hölder-continuity of $Q^+(f,f)$ has to be slightly modified.

For $w \in \mathbb{R}$, we set $\Omega := \{(u,u_*) \in \mathbb{R}^2 : u^2 + u_*^2 > w^2\}$. By a change of variables, the collision operator Q^+ can be rewritten as

$$Q^+(f,f)(w) = \int_{\mathbb{R}^2} \frac{\mathbf{1}_\Omega(u,u_*)}{\sqrt{u^2+u_*^2-w^2}}\, f(u)\, f(u_*)\, \bar{b}(u,u_*,w)\, du\, du_*,$$

where

$$\bar{b}(u,u_*,w) = b\left(\mathrm{Arccos}\left(\frac{uw + u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right)\right) + b\left(\mathrm{Arccos}\left(\frac{uw - u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right)\right).$$

Thus, for $|v| < |w| < M/2$,

$$|Q^+(f,f)(w) - Q^+(f,f)(v)| \le \int_{\mathbb{R}^2} f(u)\, f(u_*)\, \mathbf{1}_{u^2+u_*^2>w^2}\, \frac{|\bar{b}(u,u_*,w) - \bar{b}(u,u_*,v)|}{\sqrt{u^2+u_*^2-w^2}}\, du\, du_* + \|f\|_{L^\infty}^2\, \|b\|_{L^\infty} \int_{\mathbb{R}^2} \left| \frac{\mathbf{1}_{u^2+u_*^2>w^2}}{\sqrt{u^2+u_*^2-w^2}} - \frac{\mathbf{1}_{u^2+u_*^2>v^2}}{\sqrt{u^2+u_*^2-v^2}} \right| du\, du_*.$$

The second integral in the right hand side can be handled as in [9, Lemma 4]. Therefore, we only consider the first one. Since b is lipschitz continuous, we have

$$\begin{aligned}
&\left| b\left(\mathrm{Arccos}\left(\frac{uw - u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right)\right) - b\left(\mathrm{Arccos}\left(\frac{uv - u_*\sqrt{u^2+u_*^2-v^2}}{u^2+u_*^2}\right)\right)\right| \\
&\qquad \le C\left|\mathrm{Arccos}\left(\frac{uw - u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right) - \mathrm{Arccos}\left(\frac{uv - u_*\sqrt{u^2+u_*^2-v^2}}{u^2+u_*^2}\right)\right| \\
&\qquad \le C\left|\int_v^w \frac{u\sqrt{u^2+u_*^2-x^2}+u_*x}{|u\sqrt{u^2+u_*^2-x^2}+u_*x|}\ \frac{dx}{\sqrt{u^2+u_*^2-x^2}}\right| \\
&\qquad \le C\left|\int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}}\right|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left| b\left(\mathrm{Arccos}\left(\frac{uw + u_*\sqrt{u^2+u_*^2-w^2}}{u^2+u_*^2}\right)\right) - b\left(\mathrm{Arccos}\left(\frac{uv + u_*\sqrt{u^2+u_*^2-v^2}}{u^2+u_*^2}\right)\right)\right| \\
&\qquad\qquad\qquad\qquad \le C\left|\int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}}\right|.
\end{aligned}$$

We set $G = \{(u,u_*) \in \Omega : |u| \le M,\ |u_*| \le M\}$ and $B = \Omega \setminus G$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(u)\, f(u_*)\, \mathbf{1}_{u^2+u_*^2>w^2}\ \frac{|\bar b(u,u_*,w) - \bar b(u,u_*,v)|}{\sqrt{u^2+u_*^2-w^2}}\ du\, du_* \\
&\qquad \le 2\,C \int_{\mathbb{R}^2} f(u)\, f(u_*)\ \frac{\mathbf{1}_G}{\sqrt{u^2+u_*^2-w^2}}\ \left|\int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}}\right| du\, du_* \\
&\qquad + 2\,C \int_{\mathbb{R}^2} f(u)\, f(u_*)\ \frac{\mathbf{1}_B}{\sqrt{u^2+u_*^2-w^2}}\ \left|\int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}}\right| du\, du_*.
\end{aligned}$$

But, for $(u,u_*) \in B$, we have $u^2+u_*^2-w^2 \ge 3\,M^2/4$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(u)\, f(u_*)\ \frac{\mathbf{1}_B}{\sqrt{u^2+u_*^2-w^2}}\ \left|\int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}}\right| du\, du_* \\
&\qquad \le \int_{\mathbb{R}^2} f(u)\, f(u_*)\ \frac{\mathbf{1}_B}{u^2+u_*^2-w^2}\ |w-v|\, du\, du_* \le \frac{4}{3M^2}\ \|f\|_{L^1}^2\, |w-v|.
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\left|\int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}}\right| &\le \frac{1}{(u^2+u_*^2-w^2)^{1/4}}\ |w-v|^{1/2} \left|\int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}}\right|^{1/2} \\
&\le \frac{\sqrt{\pi}}{(u^2+u_*^2-w^2)^{1/4}}\ |w-v|^{1/2},
\end{aligned}$$

we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(u)\, f(u_*)\, \frac{\mathbf{1}_G}{\sqrt{u^2+u_*^2-w^2}} \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right| du\, du_* \\
&\quad \le \sqrt{\pi}\, |w-v|^{1/2} \int_{\mathbb{R}^2} f(u)\, f(u_*)\, \frac{\mathbf{1}_G}{(u^2+u_*^2-w^2)^{3/4}}\, du\, du_* \\
&\quad \le \sqrt{\pi}\, |w-v|^{1/2} \left(\int_{\mathbb{R}^2} f(u)^p\, f(u_*)^p\, du\, du_* \right)^{1/p} \left(\int_{\mathbb{R}^2} \frac{\mathbf{1}_G}{(u^2+u_*^2-w^2)^{3q/4}}\, du\, du_* \right)^{1/q}.
\end{aligned}$$

For $q = 5/4$ and $p = 5$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(u) f(u_*)\, \frac{\mathbf{1}_G}{\sqrt{u^2+u_*^2-w^2}} \left| \int_v^w \frac{dx}{\sqrt{u^2+u_*^2-x^2}} \right| du\, du_* \\
&\qquad\qquad \le \sqrt{\pi}\, (16\pi)^{4/5}\, (2M^2)^{1/20}\, \|f\|_{L^\infty}^{8/5}\, |w-v|^{1/2}.
\end{aligned}$$

□

Lemma 7 *Let $g \in L^1([0,\infty))$ be a nonnegative function satisfying $g(v) \le C(1+(\log v)^2)$ for some constant $C > 0$. For $v > 0$, we set*

$$F(v) = v^\gamma \int_v^\infty \frac{1}{w^{\gamma+1}}\, g(w)\, dw.$$

Then, for $\gamma > 0$, $F(v) \le C(1+(\log v)^2)$ and, if we also assume that g is Hölder-continuous in $v \in [0,\infty)$ then F belongs to $\mathcal{C}([0,\infty))$.

Proof of Theorem 2. Let $f_0 \in L^1(\mathbb{R})$ be a nonnegative function that has finite moments of any order and that satisfies

$$\int f_0(v)\, dv = 1, \qquad \int v\, f_0(v)\, dv = \zeta \qquad \text{and} \qquad \int v^2\, f_0(v)\, dv = 1.$$

We consider the associated sequence $(f_n)_{n\in\mathbb{N}}$ generated by the iteration (10). As stated in Remark 4, up to an extraction, this sequence converges to a measure μ. We deduce from Lemma 3 that μ satisfies

$$\int_{\mathbb{R}} \mu(dv) = 1, \qquad \int_{\mathbb{R}} v\, \mu(dv) = \zeta \qquad \text{and} \qquad \int_{\mathbb{R}} v^2\, \mu(dv) = 1.$$

This measure has a nonnegative density $f \in L^1(\mathbb{R})$. By induction, we infer from (11) that each moment of f_n is bounded, independently of n and then that each moment of f is bounded. We now pass to the limit in (10). Since $v \mapsto \int_{\mathbb{R}} \overline{\Lambda}(\psi)(v,v_*)\, \mu(dv_*)$ is a bounded and continuous function, we have

$$\int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v,v_*)\, \mu(dv_*)\, f_n(v)\, dv \longrightarrow \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v,v_*)\, \mu(dv_*)\, \mu(dv),$$

as $n \to \infty$. Then, since $\int_{\mathbb{R}} f_n(v)\, v^2\, dv = \int_{\mathbb{R}} v^2\, \mu(dv) = 1$ and $(v,v_*) \mapsto \overline{\Lambda}(\psi)(v,v_*)$ is a continuous function, we deduce that

$$\left| \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v,v_*)\, f_n(v_*)\, f_n(v)\, dv_*\, dv - \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v,v_*)\, \mu(dv_*)\, f_n(v)\, dv \right| \longrightarrow 0,$$

as $n \to \infty$. It then follows readily that

$$\int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v,v_*)\, f_n(v_*)\, f_n(v)\, dv_*\, dv \longrightarrow \int_{\mathbb{R}^2} \overline{\Lambda}(\psi)(v,v_*)\, \mu(dv_*)\, \mu(dv),$$

as $n \to \infty$. Since $Q_+(f,f) \in L^1(\mathbb{R})$, $\mathcal{A}(f)$ is continuous on $\mathbb{R}\backslash\{\kappa\}$. The proof of the first statement of Theorem 2 is now complete. Let us consider the case $\gamma > 0$. Let $k \in \mathbb{N}$. By definition of $\mathcal{A}$ and Q^+, we have

$$f_k(v) \leq C|v-\kappa|^{-1}, \qquad v \in \mathbb{R},$$

where C denotes some constant that only depends on E and b. Lemma 5 implies that

$$Q^+(f_k,f_k)(v) \leq C(1 + (\log|v-\kappa|)^2),$$

in a neighbourhood of $v = \kappa$, where C only depends on E and b. We now deduce from Lemma 7 that $f_{k+1}(v) \leq C(1+(\log|v-\kappa|)^2)$. By Lemma 6, $Q^+(f_{k+1}, f_{k+1})$ is thus bounded in a neighbourhood of $v = \kappa$. It follows from Lemma 7 that f_{k+2} is bounded. We infer from Lemma 6 that $Q^+(f_{k+2}, f_{k+2})$ is locally Hölder-continuous. Finally, Lemma 7 implies that f_{k+3} is a continuous function. Since these estimates do not depend on k, they also hold for the limit f. □

3 Non-cutoff case

We now consider (5), (2) when the cross section b satisfies (3) and prove Theorem 1.

3.1 Existence

We first investigate the existence of a weak solution to (5), (2) and prove the following theorem.

Theorem 8 *Assume that b satisfies (3). For all field strengths $E > 0$, there exists a weak solution f to (5), (2), in the sense of Definition 1, that satisfies (7) and such that moments of any order of f are finite.*

Proof. For $n \in \mathbb{N}$, we set $b_n = b \wedge n$, where, for every $c, d \in \mathbb{R}$, $c \wedge d$ denotes the minimum value of c and d. We deal with the associated stationary equation

$$E\,\frac{d}{dv}((1-\zeta_n v)f_n(v)) = Q_n(f_n,f_n)(v), \qquad v \in \mathbb{R}, \tag{12}$$

where $\zeta_n = \int_{\mathbb{R}} v f_n(v)\, dv$ and

$$Q_n(f_n,f_n)(v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} (f_n(v')f_n(v'_*) - f_n(v)f_n(v_*))\, b_n(\theta)\, d\theta\, dv_*.$$

As previously, we obtain that

$$\zeta_n = \frac{\sqrt{K_n^2 + 4E^2} - K_n}{2E},$$

with $K_n := \int_{-\pi}^{\pi} (1-\cos\theta)\, b_n(\theta)\, d\theta$. We set $\kappa_n = 1/\zeta_n$ and

$$\gamma_n = \frac{\|b_n\|_{L^1}}{E\,\zeta_n} - 1.$$

Then, any solution to (12) with $\int_{\mathbb{R}} f_n(v)\, dv = 1$ and $\int_{\mathbb{R}} v^2\, f_n(v)\, dv = 1$ satisfies

$$f_n = \mathcal{A}_n(f_n), \tag{13}$$

where

$$\mathcal{A}_n(f_n)(v) = \frac{\gamma_n + 1}{\|b_n\|_{L^1}}\, |v-\kappa_n|^{\gamma_n} \begin{cases} \displaystyle\int_{-\infty}^{v} \frac{1}{|w-\kappa_n|^{\gamma_n+1}}\, Q_n^+(f_n,f_n)(w)\, dw & (v<\kappa_n) \\ \displaystyle\int_{v}^{\infty} \frac{1}{|w-\kappa_n|^{\gamma_n+1}}\, Q_n^+(f_n,f_n)(w)\, dw & (v>\kappa_n) \end{cases}$$

with

$$Q_n^+(f_n,f_n)(v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} f_n(v')\, f_n(v'_*)\, b_n(\theta)\, d\theta\, dv_*.$$

The existence of a solution f_n to (13) satisfying $\int_{\mathbb{R}} f_n(v)\, dv = 1$ and $\int_{\mathbb{R}} v^2\, f_n(v)\, dv = 1$ follows from Theorem 2. Moreover, since γ_n goes to infinity, there exists n_0 such that, for each $n \geq n_0$, γ_n is positive. Then, for each $n \geq n_0$, the functions f_n and $Q_n^+(f_n,f_n)$ are continuous. Thus, for each $n \geq n_0$, $f_n \in \mathcal{C}^1(\mathbb{R}\backslash\{\kappa_n\})$ and, for $v \neq \kappa_n$, we have

$$\frac{d}{dv}((v-\kappa_n)f_n(v)) = -\frac{\kappa_n}{E}\, Q_n(f_n,f_n)(v).$$

For $\psi \in \mathcal{D}(\mathbb{R})$, we deduce that

$$\int_{\mathbb{R}} (v-\kappa_n)\, f_n(v)\, \psi'(v)\, dv = \frac{\kappa_n}{E} \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b_n(\theta)\, d\theta\, f_n(v)\, f_n(v_*)\, dv\, dv_*. \tag{14}$$

Since f_n has finite moments of any order, classical truncation argument ensures that, for every integer $m \geq 3$,

$$\left(m + \frac{\kappa_n}{E} \int_{-\pi}^{\pi} (1 - \cos^m\theta - \sin^m\theta)\, b_n(\theta)\, d\theta\right) \int_{\mathbb{R}} v^m f_n(v)\, dv = m\,\kappa_n \int_{\mathbb{R}} v^{m-1} f_n(v)\, dv$$
$$+ \frac{\kappa_n}{E} \sum_{k=1}^{[(m-1)/2]} \binom{m}{2k} \int_{-\pi}^{\pi} (\cos\theta)^{m-2k} (\sin\theta)^{2k}\, b_n(\theta)\, d\theta \int_{\mathbb{R}} v^{2k} f_n(v)\, dv \int_{\mathbb{R}} v^{m-2k} f_n(v)\, dv,$$

where $[a]$ denotes the integer part of $a \in \mathbb{R}$. It then follows by induction that, for each $m \in \mathbb{N}$, there exists a constant C_m independent of n such that

$$\int_{\mathbb{R}} v^m f_n(v)\, dv \leq C_m. \tag{15}$$

Consequently, $(f_n)_{n\in\mathbb{N}}$ is a tight set of unit measures and, by weak compactness, there exists a measure μ such that, up to an extraction, $f_n \rightharpoonup \mu$. Then, we have

$$\int_{\mathbb{R}} \mu(dv) = 1, \qquad \int_{\mathbb{R}} v^2 \mu(dv) = 1 \qquad \text{and} \qquad \int_{\mathbb{R}} v^m \mu(dv) \leq C_m.$$

Moreover, the measure μ has a nonnegative density $f \in L^1(\mathbb{R})$.

Let us now pass to the limit in (14). We set, for $(v, v_*) \in \mathbb{R}^2$,

$$\begin{aligned}
\Xi_n(v, v_*) &= \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b_n(\theta)\, d\theta \\
&= \frac{1}{2}\, v_*^2 \int_{-\pi}^{\pi} \int_{-1}^{1} (1 - |r|)\, \psi''(v\cos\theta + r v_* \sin\theta)\, dr \sin^2\theta\, b_n(\theta)\, d\theta \\
&\quad - v \int_{-\pi}^{\pi} \int_{0}^{1} \psi'(v + r v\, (\cos\theta - 1))\, dr\, (1 - \cos\theta)\, b_n(\theta)\, d\theta,
\end{aligned}$$

and

$$\begin{aligned}
\Xi(v, v_*) &= \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b(\theta)\, d\theta \\
&= \frac{1}{2}\, v_*^2 \int_{-\pi}^{\pi} \int_{-1}^{1} (1 - |r|)\, \psi''(v\cos\theta + r v_* \sin\theta)\, dr \sin^2\theta\, b(\theta)\, d\theta \\
&\quad - v \int_{-\pi}^{\pi} \int_{0}^{1} \psi'(v + r v\, (\cos\theta - 1))\, dr\, (1 - \cos\theta)\, b(\theta)\, d\theta.
\end{aligned}$$

Then, we deduce that

$$\begin{aligned}
|\, \Xi_n(v, v_*) - \Xi(v, v_*)\, | &\leq 2\, v_*^2\, \|\psi''\|_{L^\infty} \int_0^{n^{-1/(1+\alpha)}} \sin^2\theta\, b(\theta)\, d\theta \\
&\quad + 2\, |v|\, \|\psi'\|_{L^\infty} \int_0^{n^{-1/(1+\alpha)}} (1 - \cos\theta)\, b(\theta)\, d\theta, \qquad (16)
\end{aligned}$$

and

$$|\, \Xi_n(v, v_*)\, | \leq C\, \|\psi\|_{W^{2,\infty}}\, (v_*^2 + |v|), \qquad |\, \Xi(v, v_*)\, | \leq C\, \|\psi\|_{W^{2,\infty}}\, (v_*^2 + |v|). \qquad (17)$$

We have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \Xi_n(v, v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v, v_*)\, \mu(dv)\, \mu(dv_*) \right| \\
&\qquad \leq \left| \int_{\mathbb{R}^2} (\Xi_n(v, v_*) - \Xi(v, v_*))\, f_n(v)\, f_n(v_*)\, dv\, dv_* \right| \qquad (18) \\
&\qquad + \left| \int_{\mathbb{R}^2} \Xi(v, v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v, v_*)\, \mu(dv)\, \mu(dv_*) \right|. \qquad (19)
\end{aligned}$$

Estimates (15) and (16) ensure that the integral (18) tends to 0 as $n \to +\infty$. It thus remains to consider (19). Let $\eta \in \mathcal{C}^\infty(\mathbb{R})$ such that $\eta(v) = 1$ if $|v| \leq 1$ and $\eta(v) = 0$ if $|v| \geq 2$. Denote by η_R the function defined by $\eta_R(v) = \eta(v/R)$, for every $v \in \mathbb{R}$. Then,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \Xi(v, v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v, v_*)\, \mu(dv)\, \mu(dv_*) \right| \\
&\leq \left| \int_{\mathbb{R}^2} \Xi(v, v_*)\, \eta_R(v)\, \eta_R(v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v, v_*)\, \eta_R(v)\, \eta_R(v_*)\, \mu(dv)\, \mu(dv_*) \right| \\
&+ \int_{\mathbb{R}^2} (1 - \eta_R(v_*))\, |\, \Xi(v, v_*)|\, f_n(v)\, f_n(v_*)\, dv\, dv_* + \int_{\mathbb{R}^2} (1 - \eta_R(v_*))\, |\, \Xi(v, v_*)|\, \mu(dv)\, \mu(dv_*) \\
&+ \int_{\mathbb{R}^2} (1 - \eta_R(v))\, |\, \Xi(v, v_*)|\, f_n(v)\, f_n(v_*)\, dv\, dv_* + \int_{\mathbb{R}^2} (1 - \eta_R(v))\, |\, \Xi(v, v_*)|\, \mu(dv)\, \mu(dv_*).
\end{aligned}$$

We now infer from (15) and (17) that

$$\left|\int_{\mathbb{R}^2} \Xi(v,v_*)\, f_n(v)\, f_n(v_*)\, dv\, dv_* - \int_{\mathbb{R}^2} \Xi(v,v_*)\, \mu(dv)\, \mu(dv_*)\right|$$

$$\leq \left|\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, f_n(v_*)\, dv_* - \int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, \mu(dv_*)\right) f_n(v)\, dv\right| \quad (20)$$

$$+ \left|\int_{\mathbb{R}^2} \Xi(v,v_*)\, \eta_R(v_*)\, \mu(dv_*)\, \eta_R(v)\, f_n(v)\, dv - \int_{\mathbb{R}^2} \Xi(v,v_*)\, \eta_R(v_*)\, \mu(dv_*)\, \eta_R(v)\, \mu(dv)\right| \quad (21)$$

$$+ 2\, C\, \|\psi\|_{W^{2,\infty}}\, \frac{3+C_4}{R}.$$

Since $(v,v_*) \mapsto \eta_R(v)\, \eta_R(v_*)\, \Xi(v,v_*)$ is a compactly supported continuous function,

$$\int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, f_n(v_*)\, dv_* \longrightarrow \int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v)\, \eta_R(v_*)\, \mu(dv_*),$$

uniformly in v as $n \to +\infty$. Consequently, (20) tends to 0 as $n \to +\infty$. Moreover, the function $v \mapsto \int_{\mathbb{R}} \Xi(v,v_*)\, \eta_R(v_*)\, \mu(dv_*)\, \eta_R(v)$ belongs to $\mathcal{C}_b(\mathbb{R})$ and thus (21) also tends to 0 as $n \to +\infty$. Finally, we obtain, for every $\psi \in \mathcal{D}(\mathbb{R})$,

$$\int_{\mathbb{R}} (v-\kappa)\, \psi'(v)\, \mu(dv) = \frac{\kappa}{E} \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} (\psi(v') - \psi(v))\, b(\theta)\, d\theta\, \mu(dv)\, \mu(dv_*),$$

where

$$\kappa = \frac{1}{\zeta} = \frac{2E}{\sqrt{K^2+4E^2} - K} \qquad \text{with} \qquad K := \int_{-\pi}^{\pi} (1-\cos\theta)\, |\theta|^{-1-\alpha}\, d\theta. \quad (22)$$

Therefore, the density f of μ satisfies (6) for every $\psi \in \mathcal{D}(\mathbb{R})$ and, by density, for every $\psi \in \mathcal{C}^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$. □

3.2 Smoothness

We now show the following theorem.

Theorem 9 *Assume that b satisfies (3). For all field strengths $E > 0$, if f is a weak solution to (5), (2), in the sense of Definition 1, then $f \in \mathcal{C}^\infty(\mathbb{R})$.*

We consider here the Fourier transform $\hat{f}$ of f defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-iv\xi} f(v)\, dv, \qquad \xi \in \mathbb{R}.$$

By (6), $\hat{f}$ satisfies

$$\hat{f}'(\xi) + i\kappa \hat{f}(\xi) = \frac{\kappa}{E\,\xi} \int_{-\pi}^{\pi} \left(\hat{f}(\xi\cos\theta)\, \hat{f}(\xi\sin\theta) - \hat{f}(0)\, \hat{f}(\xi)\right)\, b(\theta)\, d\theta, \quad (23)$$

where $\kappa = 1/\zeta$. We set, for every $\xi \in \mathbb{R}$,

$$q_1(\xi) = -\frac{1}{2\,E} \int_{-\pi}^{\pi} \left(\hat{f}(\xi\sin\theta) + \hat{f}(-\xi\sin\theta) - 2\hat{f}(0)\right)\, b(\theta)\, d\theta,$$

$$q_2(\xi) = \frac{\kappa}{E} \int_{-\pi}^{\pi} \hat{f}(\xi\sin\theta) \left(\hat{f}(\xi\cos\theta) - \hat{f}(\xi)\right)\, b(\theta)\, d\theta.$$

Then, (23) reads

$$\hat{f}'(\xi) + \kappa\left(i + \frac{q_1(\xi)}{\xi}\right)\hat{f}(\xi) = \frac{q_2(\xi)}{\xi}, \qquad \xi \in \mathbb{R}. \tag{24}$$

Lemma 10 *There exists $M \geq 1$ such that, for $|\xi| \geq M$,*

$$q_1(\xi) \geq D_f\,|\xi|^\alpha \qquad \textit{with} \quad D_f = \frac{1}{4\,E}\int_{\mathbb{R}} f(v)\,|v|^\alpha dv \int_0^\infty \sin^2(y/2)|y|^{-1-\alpha}dy > 0, \tag{25}$$

$$|q_2(\xi)| \leq G_f\,|\xi|^{\alpha/2} \qquad \textit{with} \quad G_f = \frac{3\,\kappa}{E}\int_{\mathbb{R}} f(v)\,|v|^{\alpha/2}dv \int_0^\infty \left|e^{2iy^2}-1\right|\,|y|^{-1-\alpha}dy + 20\,\frac{\kappa}{E}. \tag{26}$$

Proof. The function q_1 may be written under the form

$$q_1(\xi) = \frac{2}{E}\int_{\mathbb{R}} f(v)\int_{-\pi}^{\pi}\sin^2\left(\frac{v\,\xi\,\sin\theta}{2}\right)|\theta|^{-1-\alpha}\,d\theta\,dv.$$

Thus, $q_1(\xi)$ is real and

$$q_1(\xi) \geq \frac{4}{E}\int_{\mathbb{R}} f(v)\int_0^{\pi/2}\sin^2\left(\frac{v\,\xi\,\sin\theta}{2}\right)|\theta|^{-1-\alpha}\,d\theta\,dv.$$

The successive changes of variables $x = \sin\theta$ and $y = |v\,\xi|\,x$ lead to

$$\begin{aligned} q_1(\xi) &\geq \frac{4}{E}\int_{\mathbb{R}} f(v)\int_0^1 \sin^2\left(\frac{v\,\xi\,x}{2}\right)|2x|^{-1-\alpha}\,dx\,dv \\ &\geq \frac{2^{1-\alpha}}{E}\,|\xi|^\alpha\int_{\mathbb{R}} f(v)\,|v|^\alpha\int_0^{|v\xi|}\sin^2(y/2)\,|y|^{-1-\alpha}\,dy\,dv. \end{aligned}$$

Since we have

$$\int_{\mathbb{R}} f(v)\,|v|^\alpha\int_0^{|v\xi|}\sin^2(y/2)\,|y|^{-1-\alpha}\,dy\,dv \longrightarrow \int_{\mathbb{R}} f(v)\,|v|^\alpha\,dv\int_0^\infty\sin^2(y/2)\,|y|^{-1-\alpha}\,dy,$$

as $|\xi| \to \infty$, we deduce that there exists $M \geq 0$ such that (25) holds for $|\xi| \geq M$. Since

$$\begin{aligned} \int_{\mathbb{R}} f(v)\,|v|^\alpha\,dv &\geq R^{\alpha-2}\int_{|v|\leq R} f(v)\,|v|^2\,dv \\ &\geq R^{\alpha-2}\left(\int_{\mathbb{R}} f(v)\,|v|^2\,dv - \int_{|v|\geq R} f(v)\,|v|^2\,dv\right), \end{aligned}$$

the constant D_f is positive. As for q_2, we have

$$\begin{aligned} |q_2(\xi)| &\leq \frac{\kappa}{E}\int_{\mathbb{R}}\int_{-\pi}^{\pi} f(v)\left|e^{-iv\xi(\cos\theta-1)}-1\right|\,|\theta|^{-1-\alpha}\,d\theta\,dv \\ &\leq \frac{2\,\kappa}{E}\int_{\mathbb{R}}\int_0^{\pi/2} f(v)\left|e^{2iv\xi\sin^2(\theta/2)}-1\right|\,|\theta|^{-1-\alpha}\,d\theta\,dv + 4\,(2/\pi)^\alpha\,\frac{\kappa}{E}. \end{aligned}$$

The successive changes of variables $u = \sin(\theta/2)$ and $y = \sqrt{|v\,\xi|}\,u$ imply that

$$\begin{aligned} \int_0^{\pi/2}\left|e^{2iv\xi\sin^2(\theta/2)}-1\right|\,|\theta|^{-1-\alpha}\,d\theta &\leq 2^{1/2-\alpha}\int_0^{1/\sqrt{2}}\left|e^{2i|v\xi|u^2}-1\right|\,|u|^{-1-\alpha}\,du \\ &\leq 2^{1/2-\alpha}\,|\xi|^{\alpha/2}\,|v|^{\alpha/2}\int_0^{\sqrt{|v\xi|/2}}\left|e^{2iy^2}-1\right|\,|y|^{-1-\alpha}\,dy \\ &\leq 2^{1/2}\,|\xi|^{\alpha/2}\,|v|^{\alpha/2}\int_0^\infty\left|e^{2iy^2}-1\right|\,|y|^{-1-\alpha}dy. \end{aligned}$$

□

Lemma 11 *Denote by M the constant given by Lemma 10. There exists some constant $C > 0$ such that, for $|\xi| \geq M$,*

$$|\hat{f}(\xi)| \leq C\,|\xi|^{-\alpha/2} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq \begin{cases} C\,|\xi|^{-\alpha/2}, & \alpha \leq 1, \\ C\,|\xi|^{\alpha/2-1}, & \alpha \geq 1. \end{cases} \tag{27}$$

Proof. We deduce from (24) that, for $\xi \geq M$,

$$\hat{f}(\xi) = \hat{f}(M)\, e^{-i\kappa(\xi-M)}\, e^{-\int_M^\xi \kappa \frac{q_1(u)}{u} du} + e^{-i\kappa\xi} \int_M^\xi e^{i\kappa\eta}\, e^{-\int_\eta^\xi \kappa \frac{q_1(u)}{u} du}\, \frac{q_2(\eta)}{\eta}\, d\eta.$$

Lemma 10 then implies that

$$|\hat{f}(\xi)| \leq C\, e^{-D_f \kappa |\xi|^\alpha/\alpha} + G_f\, e^{-D_f \kappa |\xi|^\alpha/\alpha} \int_M^{|\xi|} e^{D_f \kappa \eta^\alpha/\alpha}\, \eta^{\alpha/2-1}\, d\eta. \tag{28}$$

Similarly, we obtain the same estimate for $\xi \leq -M$. We now check that $|\xi|^{\alpha/2}\,|\hat{f}(\xi)|$ is bounded for $|\xi| \geq M$. By (28), it suffices to show that $F_1(\xi)/F_2(\xi)$ is bounded, where

$$F_1(\xi) = \int_M^{|\xi|} e^{D_f \kappa \eta^\alpha/\alpha}\, \eta^{\alpha/2-1}\, d\eta \qquad \text{and} \qquad F_2(\xi) = |\xi|^{-\alpha/2}\, e^{D_f \kappa |\xi|^\alpha/\alpha}.$$

Since $F_1'(\xi)/F_2'(\xi)$ tends to a constant as $|\xi| \to \infty$, we infer from l'Hospital's rule that $F_1(\xi)/F_2(\xi)$ tends to a constant as $|\xi| \to \infty$ and thus is bounded for $|\xi| \geq M$. This completes the proof of the first inequality of (27).

Proceeding as in the proof of (25), we have, for some $C > 0$,

$$|q_1(\xi)| \leq C\,|\xi|^\alpha, \qquad |\xi| \geq 1, \tag{29}$$

which together with equations (24), (26) and the first inequality of (27) implies that

$$|\hat{f}'(\xi)| \leq C\,(|\xi|^{-\alpha/2} + |\xi|^{\alpha/2-1}), \qquad |\xi| \geq M,$$

and, consequently, the second estimate of (27). □

Lemma 12 *Denote by M the constant given by Lemma 10. We assume that there exists $\delta \geq 0$ and $C > 0$ such that, for every $|\xi| \geq M$,*

$$|\hat{f}(\xi)| \leq C\,|\xi|^{-\delta} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq C\,|\xi|^{-\delta}, \tag{30}$$

if $\alpha \in (0,1]$ and

$$|\hat{f}(\xi)| \leq C\,|\xi|^{-\delta} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq C\,|\xi|^{-\delta+\alpha-1}, \tag{31}$$

if $\alpha \in (1,2)$. Then, for every $\varepsilon <$ min $(\alpha/2, 1-\alpha/2)$, there exists a constant $C(\varepsilon) > 0$ such that, for $|\xi| \geq M$,

$$|\hat{f}(\xi)| \leq C(\varepsilon)\,|\xi|^{-\delta-\alpha/2+\varepsilon} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq C(\varepsilon)\,|\xi|^{-\delta-\alpha/2+\varepsilon}, \tag{32}$$

if $\alpha \in (0,1]$ and

$$|\hat{f}(\xi)| \leq C(\varepsilon)\,|\xi|^{-\delta-\alpha(1-\alpha/2-\varepsilon)} \qquad \text{and} \qquad |\hat{f}'(\xi)| \leq C(\varepsilon)\,|\xi|^{-\delta-\alpha(1-\alpha/2-\varepsilon)+\alpha-1}, \tag{33}$$

if $\alpha \in (1,2)$.

Proof. We first consider the case $\alpha \in (0,1]$. Let $\varepsilon > 0$ be such that $1-\alpha/2-\varepsilon > 0$ and $\varepsilon < \alpha/2$. Then,

$$\begin{aligned} |q_2(\xi)| &\leq \frac{\kappa}{E}\,|\xi|^{\alpha/2+\varepsilon} \int_{-\pi/4}^{\pi/4} \left|\hat{f}(\xi\cos\theta) - \hat{f}(\xi)\right|^{1-\alpha/2-\varepsilon} \left|\int_0^1 \hat{f}'(\xi + u\xi(\cos\theta - 1))\,du\right|^{\alpha/2+\varepsilon} \\ &\quad \times (1-\cos\theta)^{\alpha/2+\varepsilon}\,|\theta|^{-1-\alpha}\,d\theta + \frac{\kappa}{E}\int_{|\theta\pm\pi/4|\leq\pi/2} \hat{f}(\xi\sin\theta)\,|\theta|^{-1-\alpha}\,d\theta \\ &\quad + \frac{\kappa}{E}\int_{3\pi/4\leq|\theta|\leq\pi} \left|\hat{f}(\xi\cos\theta) - \hat{f}(\xi)\right|\,|\theta|^{-1-\alpha}\,d\theta. \end{aligned} \tag{34}$$

Since $\hat{f}$ satisfies (30), we thus deduce that

$$|q_2(\xi)| \leq C\,|\xi|^{-\delta+\alpha/2+\varepsilon} + C\,|\xi|^{-\delta} \leq C\,|\xi|^{-\delta+\alpha/2+\varepsilon}, \qquad |\xi| \geq M.$$

Similarly to the proof of the first inequality of (27), we obtain

$$\begin{aligned} |\hat{f}(\xi)| &\leq C\,e^{-\kappa D_f|\xi|^\alpha/\alpha} + C\,e^{-\kappa D_f|\xi|^\alpha/\alpha}\int_M^{|\xi|} e^{\kappa D_f\eta^\alpha/\alpha}\,\eta^{-\delta+\alpha/2+\varepsilon-1}d\eta, \\ &\leq C\,|\xi|^{-\delta-\alpha/2+\varepsilon}, \end{aligned}$$

for $|\xi| \geq M$. Consequently, we infer from (24), (29) and the above estimates that

$$|\hat{f}'(\xi)| \leq C\,|\xi|^{-\delta-\alpha/2+\varepsilon}, \qquad |\xi| \geq M,$$

which completes the proof of (32).

We now turn our attention to the case $\alpha \in (1,2)$. Let $\varepsilon > 0$ be such that $1-\alpha/2-\varepsilon > 0$. Equations (34) and (31) lead to

$$|q_2(\xi)| \leq C\,|\xi|^{-\delta-\alpha(1-\alpha/2-\varepsilon)+\alpha},$$

and (33) follows as previously. □

Proof of Theorem 9. Let f be a weak solution to (5), (2). It is sufficient to prove that the Fourier transform $\hat{f}$ of f satisfies, for every $s \geq 0$,

$$\hat{f}(\xi) \leq \frac{C(s)}{1+|\xi|^s}, \qquad \xi \in \mathbb{R}, \tag{35}$$

where $C(s)$ denotes some positive constant. By induction, it follows from Lemmas 11 and 12 that, for every $s \geq 0$, there exists a constant $C'(s) > 0$ such that

$$\hat{f}(\xi) \leq C'(s)\,|\xi|^{-s}, \qquad |\xi| \geq M.$$

Since $f \in L^1(\mathbb{R})$, $\hat{f}$ is bounded and (35) holds. □

3.3 Uniqueness

We now show the following theorem.

Theorem 13 *Assume that b satisfies (3). For all field strengths $E > 0$, there is at most one weak solution to (5), (2) whose moments of any order are finite and such that $\int_{\mathbb{R}} f(v)\,dv = 1$.*

Let f be a weak solution to (5), (2) such that any moments of f are finite and $\int_{\mathbb{R}} f(v)\,dv = 1$. Then, classical truncation argument ensures that, for every integer $m \geq 2$,

$$\left(m\,\zeta E + \int_{-\pi}^{\pi} (1 - \cos^m\theta - \sin^m\theta)\,|\theta|^{-1-\alpha}\,d\theta\right) \int_{\mathbb{R}} v^m f(v)\,dv = m\,E \int_{\mathbb{R}} v^{m-1} f(v)\,dv$$
$$+ \sum_{k=1}^{[(m-1)/2]} \binom{m}{2k} \int_{-\pi}^{\pi} (\cos\theta)^{m-2k}\,(\sin\theta)^{2k}\,|\theta|^{-1-\alpha}\,d\theta \int_{\mathbb{R}} v^{2k} f(v)\,dv \int_{\mathbb{R}} v^{m-2k} f(v)\,dv, \quad (36)$$

where $\zeta = (\sqrt{K^2+4E^2} - K)/(2E)$. We set $w_m := \int_{\mathbb{R}} f(v)\,v^m\,dv/m!$, $m \in \mathbb{N}$. Then, we have

$$w_0 = 1, \qquad w_1 = \zeta \qquad \text{and} \qquad w_2 = \frac{1}{2}, \quad (37)$$

and the sequence $(w_m)_{m\in\mathbb{N}}$ satisfies

$$w_m = A_m\,w_{m-1} + \sum_{k=1}^{[(m-1)/2]} B_{m,k}\,w_{2k}\,w_{m-2k}, \qquad m \geq 3, \quad (38)$$

where

$$A_m = \frac{E}{m\,\zeta E + \int_{-\pi}^{\pi} (1 - \cos^m\theta - \sin^m\theta)\,|\theta|^{-1-\alpha}\,d\theta}, \quad (39)$$

$$B_{m,k} = \frac{\int_{-\pi}^{\pi} (\cos\theta)^{m-2k} (\sin\theta)^{2k}\,|\theta|^{-1-\alpha}\,d\theta}{m\,\zeta E + \int_{-\pi}^{\pi} (1 - \cos^m\theta - \sin^m\theta)\,|\theta|^{-1-\alpha}\,d\theta}. \quad (40)$$

Lemma 14 *The coefficients $B_{m,k}$ defined by (40) satisfy*

$$B_{m,k} \leq \frac{2}{m-1}, \qquad 1 \leq k \leq [m/2] - 1, \quad m \geq 3. \quad (41)$$

Proof. If $m = 2p$, $p \geq 2$, we have

$$\begin{aligned} 1 - (\cos\theta)^{2p} - (\sin\theta)^{2p} &= (\cos^2\theta + \sin^2\theta)^p - (\cos\theta)^{2p} - (\sin\theta)^{2p} \\ &= \sum_{k=1}^{p-1} \binom{p}{k} (\cos\theta)^{2p-2k} (\sin\theta)^{2k} \end{aligned}$$

We notice that $\binom{p}{k} \geq p$ for $1 \leq k \leq p-1$ and we thus deduce that

$$1 - (\cos\theta)^{2p} - (\sin\theta)^{2p} \geq p\,(\cos\theta)^{2p-2k} (\sin\theta)^{2k}, \qquad 1 \leq k \leq p-1,$$

whence $B_{2p,k} \leq 1/p$ and (41) holds when m is even.

For $m = 2p+1$, $B_{m,k}$ reads

$$B_{2p+1,k} = \frac{\int_{-\pi}^{\pi} (\cos\theta)^{2p+1-2k} (\sin\theta)^{2k}\,|\theta|^{-1-\alpha}\,d\theta}{(2p+1)\,\zeta E + \int_{-\pi}^{\pi} (1 - (\cos\theta)^{2p+1})\,|\theta|^{-1-\alpha}\,d\theta}.$$

We have

$$1 - (\cos\theta)^{2p+1} \geq 1 - (\cos\theta)^{2p} \geq 1 - (\cos\theta)^{2p} - (\sin\theta)^{2p}.$$

Consequently, for $1 \le k \le p-1$, we have

$$B_{2p+1,k} \le \frac{\int_{-\pi}^{\pi} (\cos\theta)^{2p-2k} (\sin\theta)^{2k} \, |\theta|^{-1-\alpha} \, d\theta}{2p\,\zeta E + \int_{-\pi}^{\pi} (1 - (\cos\theta)^{2p} - (\sin\theta)^{2p}) \, |\theta|^{-1-\alpha} \, d\theta} = B_{2p,k} \le \frac{1}{p},$$

which completes the proof of (41). $\square$

Lemma 15 *The coefficients A_m defined by (39) satisfy*

$$A_{2p} \le \frac{1}{2}, \qquad p \ge 2. \tag{42}$$

Proof. It suffices to show that $A_4 \le 1/2$. We claim that

$$\int_{-\pi}^{\pi} (1 - \cos^4\theta - \sin^4\theta) \, |\theta|^{-1-\alpha} \, d\theta \ge \frac{3}{10} \int_{-\pi}^{\pi} (1 - \cos\theta) \, |\theta|^{-1-\alpha} \, d\theta. \tag{43}$$

Then, we obtain

$$A_4 \le \frac{E/K}{4\,\zeta E/K + 3/10},$$

where K is given by (22). Setting $x = E/K$, it is easily checked that

$$\frac{x}{2\sqrt{1+4x^2} - 17/10} \le \frac{1}{2}, \qquad x \in \mathbb{R}_+,$$

and thus that $A_4 \le 1/2$.

It now remains to prove (43). We point out that

$$1 - \cos^4\theta - \sin^4\theta = \frac{1}{2} \, \sin^2(2\,\theta), \qquad \theta \in [-\pi, \pi].$$

Consequently,

$$\int_{-\pi}^{\pi} (1 - \cos^4\theta - \sin^4\theta) \, |\theta|^{-1-\alpha} \, d\theta = \int_0^{\pi} \theta^{-1-\alpha} \, \sin^2(2\theta) \, d\theta = 2^{\alpha} \int_0^{2\pi} \theta^{-1-\alpha} \, \sin^2\theta \, d\theta.$$

Moreover, we have

$$\int_{-\pi}^{\pi} (1 - \cos\theta) \, |\theta|^{-1-\alpha} \, d\theta = 4 \int_0^{\pi} \theta^{-1-\alpha} \, \sin^2(\theta/2) \, d\theta = 2^{2-\alpha} \int_0^{\pi/2} \theta^{-1-\alpha} \, \sin^2\theta \, d\theta.$$

We then deduce that

$$\frac{\int_{-\pi}^{\pi} (1 - \cos^4\theta - \sin^4\theta) \, |\theta|^{-1-\alpha} \, d\theta}{\int_{-\pi}^{\pi} (1 - \cos\theta) \, |\theta|^{-1-\alpha} \, d\theta} = 4^{\alpha-1} \left(1 + \frac{\int_{\pi/2}^{2\pi} \theta^{-1-\alpha} \, \sin^2\theta \, d\theta}{\int_0^{\pi/2} \theta^{-1-\alpha} \, \sin^2\theta \, d\theta} \right).$$

But,

$$\int_{\pi/2}^{2\pi} \theta^{-1-\alpha} \, \sin^2\theta \, d\theta \ge (2\pi)^{-1-\alpha} \int_{\pi/2}^{2\pi} \sin^2\theta \, d\theta \ge (2\pi)^{-1-\alpha} \frac{3\pi}{4},$$

and, since $\alpha < 2$,

$$\int_0^{\pi/2} \theta^{-1-\alpha} \, \sin^2\theta \, d\theta \le \int_0^{\pi/2} \theta^{1-\alpha} \, d\theta \le \left(\frac{\pi}{2}\right)^{2-\alpha} \frac{1}{2-\alpha}.$$

Thus,

$$\frac{\int_{\pi/2}^{2\pi} \theta^{-1-\alpha}\ \sin^2\theta\, d\theta}{\int_0^{\pi/2} \theta^{-1-\alpha}\ \sin^2\theta\, d\theta} \geq (2-\alpha)\, 4^{-\alpha}\, \frac{3}{2\pi^2},$$

and

$$\int_{-\pi}^{\pi} (1-\cos^4\theta - \sin^4\theta)\, |\theta|^{-1-\alpha}\, d\theta \geq \left(4^{\alpha-1} + (2-\alpha)\, \frac{3}{8\pi^2}\right) \int_{-\pi}^{\pi} (1-\cos\theta)\, |\theta|^{-1-\alpha}\, d\theta.$$

Since $\alpha \mapsto 4^{\alpha-1} + 3\,(2-\alpha)/(8\pi^2)$ is a non-decreasing function, we obtain

$$\int_{-\pi}^{\pi} (1-\cos^4\theta - \sin^4\theta)\, |\theta|^{-1-\alpha}\, d\theta \geq \left(\frac{1}{4} + \frac{3}{4\pi^2}\right) \int_{-\pi}^{\pi} (1-\cos\theta)\, |\theta|^{-1-\alpha}\, d\theta.$$

Therefore, (43) holds. □

Lemma 16 *The coefficients A_m and $B_{m,k}$ defined by (39) and (40) satisfy*

$$A_{2p+1} + w_1\, B_{2p+1,p} \leq \frac{1}{2} + \frac{1}{2p}, \qquad p \geq 1. \tag{44}$$

Proof. We first consider the case $p = 1$. Since ζ satisfies $\zeta^2 + (K/E)\,\zeta - 1 = 0$, we have

$$A_3 + w_1\, B_{3,1} = \zeta + \frac{2K\zeta - 2E}{3\,\zeta E + \int_{-\pi}^{\pi} (1-\cos^3\theta)\, |\theta|^{-1-\alpha}\, d\theta}.$$

Making use of the inequalities

$$1 - \cos^3\theta \geq \frac{1}{2}\,(1-\cos\theta), \qquad 1-\cos^3\theta \leq 3\,(1-\cos\theta), \qquad \theta \in [0,\pi],$$

we obtain

$$A_3 + w_1\, B_{3,1} \leq \zeta + \frac{2\,\zeta}{3\,\zeta E/K + 1/2} - \frac{2E/K}{3\,\zeta E/K + 3}.$$

Let us show that the right hand side of this inequality is less than 1. Setting $x = E/K$, this amounts to showing that

$$\frac{\sqrt{1+4x^2}-1}{2x} + \frac{2\,(\sqrt{1+4x^2}-1)}{3x\sqrt{1+4x^2} - 2x} - \frac{4x}{3\,(\sqrt{1+4x^2}+1)} \leq 1, \qquad x \in \mathbb{R}_+,$$

that is,

$$\frac{2x\,(3\sqrt{1+4x^2}+10)}{3\,(3\,\sqrt{1+4x^2}-2)(\sqrt{1+4x^2}+1)} \leq 1, \qquad x \in \mathbb{R}_+.$$

Consequently, we want to show that

$$3\,(1-2x)\sqrt{1+4x^2} + 36x^2 - 20x + 3 \geq 0, \qquad x \in \mathbb{R}_+.$$

This holds for $x \in [0, 1/2]$ since $36x^2 - 20x + 3 \geq 0$. For $x \in (1/2, \infty)$, we use the inequality $\sqrt{1+4x^2} \leq 1 + 2x$. It then suffices to check that

$$12x^2 - 10x + 3 \geq 0, \qquad x \in (1/2, \infty),$$

which completes the proof of (44) for $p = 1$.

We now consider the general case $p \geq 2$. We have

$$\cos\theta\,(\sin\theta)^{2p} \leq 1-\cos\theta, \qquad 1-(\cos\theta)^{2p+1} \geq \frac{1}{2}\,(1-\cos\theta), \qquad \theta\in(0,\pi), \quad p\geq 2.$$

Consequently,

$$A_{2p+1} + w_1\, B_{2p+1,p} \leq \frac{E/K+\zeta}{(2p+1)\,\zeta E/K + 1/2}.$$

Proving that the right hand side of this inequality is bounded from above by $1/2 + 1/(2p)$ is equivalent to showing that

$$x + \frac{\sqrt{1+4x^2}-1}{2x} \leq \frac{p+1}{4p}\,((2p+1)\sqrt{1+4x^2}-2p), \qquad x\in\mathbb{R}_+,$$

where x stands for E/K. This amounts to proving that

$$\frac{\sqrt{1+4x^2}-1}{x} + 2x + (p+1) - \left(p+\frac{3}{2}+\frac{1}{2p}\right)\sqrt{1+4x^2} \leq 0, \qquad x\in\mathbb{R}_+,$$

Since

$$\frac{\sqrt{1+4x^2}-1}{x} \leq 2x, \qquad x\in\mathbb{R}_+,$$

it suffices to check that $\varphi_p(x)\leq 0$ for $x\in\mathbb{R}_+$, $p\geq 2$, where

$$\varphi_p(x) = 4x + (p+1) - \left(p+\frac{3}{2}+\frac{1}{2p}\right)\sqrt{1+4x^2}, \qquad x\in\mathbb{R}_+, \quad p\geq 2.$$

For $x\geq 0$, we have

$$\varphi_p(x) \leq \varphi_p\left(\frac{1}{\sqrt{(p+3/2+1/(2p))^2-4}}\right) = p+1-\sqrt{\left(p+\frac{3}{2}+\frac{1}{2p}\right)^2-4}.$$

Let us check that

$$p+1-\sqrt{\left(p+\frac{3}{2}+\frac{1}{2p}\right)^2-4} \leq 0, \qquad p\geq 2. \tag{45}$$

For $p=2$, it holds. For $p\geq 3$, we have $p+1\leq\sqrt{(p+3/2)^2-4}$, which implies (45). □

Lemma 17 *The sequence $(w_m)_{m\in\mathbb{N}}$ defined by (37), (38) is nonnegative. Moreover, $(w_m)_{m\geq 2}$ is bounded from above by $1/2$.*

Proof. Let us first check that the coefficients $B_{m,k}$ are nonnegative. This is straightforward when m is even. If m is odd, an integration by parts leads to

$$\begin{aligned}\int_{-\pi}^{\pi}(\cos\theta)^{m-2k}(\sin\theta)^{2k}\,|\theta|^{-1-\alpha}\,d\theta &= \frac{2\,(\alpha+1)}{2k+1}\int_0^{\pi}(\cos\theta)^{m-2k-1}(\sin\theta)^{2k+1}\,|\theta|^{-2-\alpha}\,d\theta \\ &\quad + \frac{m-2k-1}{2k+1}\int_{-\pi}^{\pi}(\cos\theta)^{m-2(k+1)}(\sin\theta)^{2(k+1)}\,|\theta|^{-1-\alpha}\,d\theta. \end{aligned} \tag{46}$$

For $k=(m-1)/2$, we deduce that, for $m\geq 3$,

$$\int_{-\pi}^{\pi}\cos\theta(\sin\theta)^{m-1}\,|\theta|^{-1-\alpha}\,d\theta = \frac{2(\alpha+1)}{m}\int_0^{\pi}\sin^m\theta\,|\theta|^{-2-\alpha}\,d\theta,$$

is nonnegative. Since the first integral in the right hand side of (46) is nonnegative, it then follows by induction according to the decreasing values of k that (46) is nonnegative for $1 \leq k \leq [(m-1)/2]$, $m \geq 3$. It then implies that the coefficients $B_{m,k}$ are nonnegative and, then that w_m is nonnegative for every $m \in \mathbb{N}$.

We infer from Lemmas 14, 15 and 16 that the sequence $(w_m)_{m \geq 2}$ is bounded by $1/2$. $\square$

Proof of Theorem 13. Let us assume that there exist two weak solutions f and g to (5), (2) whose moments of any order are finite and such that

$$\int_{\mathbb{R}} f(v)\, dv = \int_{\mathbb{R}} g(v)\, dv = 1.$$

Then the moments of f and g both satisfy (36). We then deduce by induction that

$$u_m := \int_{\mathbb{R}} f(v)\, v^m\, dv = \int_{\mathbb{R}} g(v)\, v^m\, dv, \qquad m \in \mathbb{N}.$$

Since f has finite moments of any order, $\hat{f} \in \mathcal{C}^\infty(\mathbb{R})$ and $|\hat{f}^{(m)}(\xi)| \leq \tilde{u}_m$, where $\tilde{u}_m := \int_{\mathbb{R}} f(v)\, |v|^m\, dv$, $m \in \mathbb{N}$. Let $r \in (0,1)$ and $t \in \mathbb{R}$. The Taylor Lagrange formula reads

$$\left| \hat{f}(t+h) - \sum_{m=0}^{k} \frac{\hat{f}^{(m)}(t)}{m!}\, h^m \right| \leq \frac{\tilde{u}_{k+1}}{(k+1)!}\, h^{k+1}, \qquad h \in \mathbb{R}, \quad k \in \mathbb{N}.$$

By Lemma 17, we know that $0 \leq u_m/m! \leq 1/2$ for $m \geq 2$. Thus, $\tilde{u}_{2k}/(2k)! \leq 1/2$, $k \geq 1$. Since $|v|^{2k-1} \leq 1 + |v|^{2k}$, we deduce that

$$\frac{\tilde{u}_{2k-1}}{(2k-1)!} \leq \frac{1}{(2k-1)!} + \frac{\tilde{u}_{2k}}{(2k-1)!} \leq \frac{1}{(2k-1)!} + k, \qquad k \geq 1.$$

Consequently, we have, for every $k \in \mathbb{N}$,

$$\frac{\tilde{u}_{k+1}}{(k+1)!} \leq \frac{k+4}{2},$$

and, for $|h| \leq r$,

$$\lim_{k \to +\infty} \frac{\tilde{u}_{k+1}}{(k+1)!}\, h^{k+1} = 0$$

holds. Therefore, for $|h| \leq r$,

$$\hat{f}(t+h) = \sum_{m=0}^{\infty} \frac{\hat{f}^{(m)}(t)}{m!}\, h^m.$$

The same argument gives

$$\hat{g}(t+h) = \sum_{m=0}^{\infty} \frac{\hat{g}^{(m)}(t)}{m!}\, h^m, \qquad |h| \leq r.$$

Since $\hat{f}^{(m)}(0) = (-i)^m u_m = \hat{g}^{(m)}(0)$, we deduce that $\hat{f} = \hat{g}$ on $(-r, r)$. By bootstrap, we obtain that $\hat{f} = \hat{g}$ on $\mathbb{R}$. The proof of Theorem 9 implies that $\hat{f}$ and $\hat{g}$ both belong to $L^1(\mathbb{R})$ and the Fourier inversion theorem gives $f = g$. $\square$

4 Numerical experiments

In this section we show the results from some numerical experiments, that were carried out to illustrate the theoretical results. We solved equation (24) as an initial value problem starting at $\xi = 0$ with $\hat{f} = 1$. The equations were solved iteratively by computing first q_1 and q_2 by standard quadrature routines, and then integrating the ordinary differential equation by a Runge-Kutta method. Then finally the distribution $f(v)$ was obtained by Fast Fourier Transform. All was done using standard routines in Matlab in a relatively straight forward way.

Figure 1 shows the Fourier transform $\hat{f}$ for $\alpha = 1.0$ and $E = 3.0$; obviously, because f is real, $\Re\hat{f}$ is even and $\Im\hat{f}$ is odd. Then Figures 2 and 3 show the result of computing the inverse transform for obtaining the function f. Figures 2 and 3 illustrate repectively the influence of the force field E and of the power α in the cross section. In Figure 2, $\alpha = 1.0$ and the results for three different values of E are shown whereas the plot of the function f for $E = 5.0$ but for three different values of α is reported in Figure 3. By Theorem 1, the stationary solutions to (5) belong to $\mathcal{C}^\infty(\mathbb{R})$ for any $E > 0$ and any $\alpha \in (0, 2)$ but, at fixed α, when E increases, the stationary solution tends to concentrate at one point (see Figure 2). On the oher hand, at fixed E, the stationary solution tends to concentrate at one point as well when α decreases (see Figure 3).

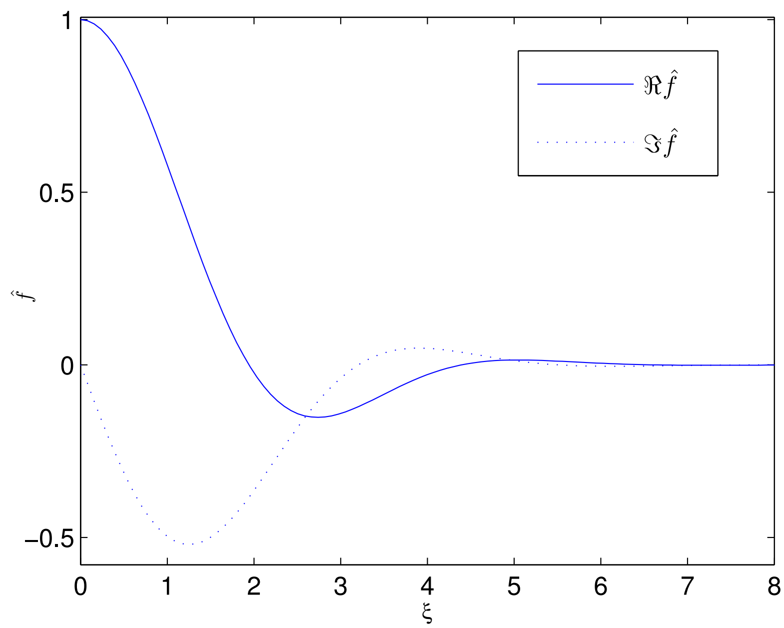

Figure 1: The Fourier transform of $f(v)$ for $E = 3.0$ and $b(\theta) = |\theta|^{-1-\alpha}$ with $\alpha = 1.0$.

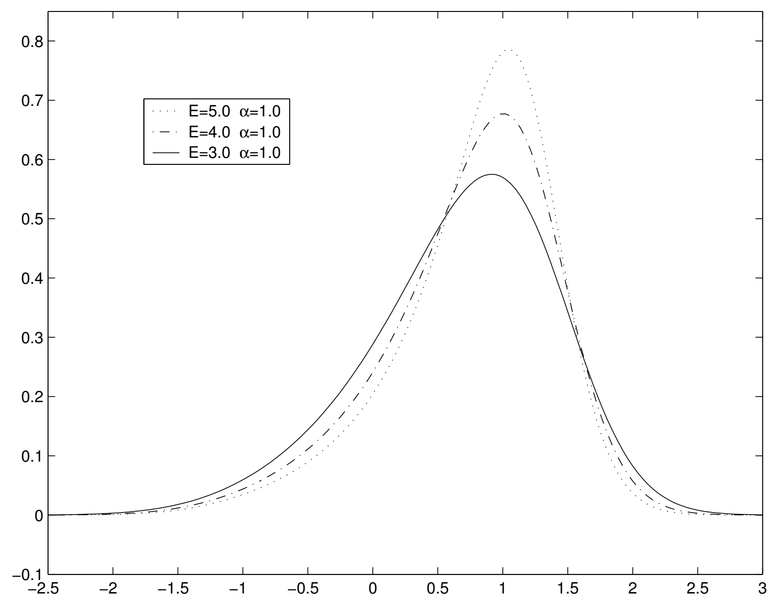

Figure 2: The solution $f(v)$ for $E = 3.0$, $E = 4.0$, $E = 5.0$ and $b(\theta) = |\theta|^{-1-\alpha}$ with $\alpha = 1.0$.

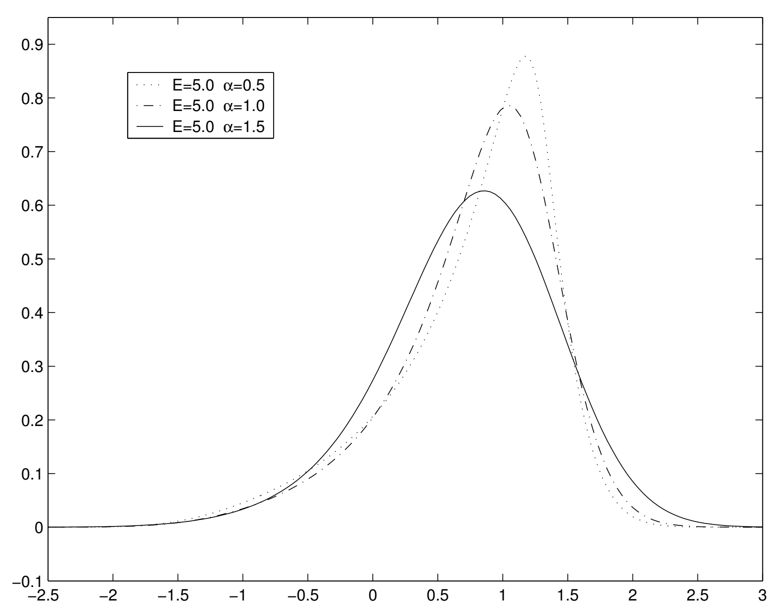

Figure 3: The solution $f(v)$ for $E = 5.0$ and $b(\theta) = |\theta|^{-1-\alpha}$ with $\alpha = 0.5, 1.0$ and 1.5.

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