Self-similar solutions to a coagulation equation

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Abstract

The existence of self-similar solutions with a finite first moment is established for the Oort-Hulst-Safronov coagulation equation when the coagulation kernel is given by $a(y,y_*) = y^\lambda + y_*^\lambda$ for some $\lambda \in (0,1)$. The corresponding self-similar profiles are compactly supported and have a discontinuity at the edge of their support.

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1 Introduction

Coagulation models provide a mean-field description of particle growth, the particles increasing their size as a consequence of successive binary mergers. Assuming the particles to be fully identified by their size, the resulting equations determine the dynamics of the size distribution function $g(t,y) \geq 0$ of particles of size $y \in \mathbb{R}_+ := (0,\infty)$ at time $t \geq 0$. Of particular interest are the predictions concerning the large time behaviour of the size distribution of the particles which can be drawn from these models. It is actually commonly expected that, for coagulation kernels with a moderate growth for large sizes, the size distribution function g should approach a mass-conserving self-similar function g_s for large times, that is,

$$g(t,y) \sim g_s(t,y) = \frac{1}{s(t)^2}\, \xi\left(\frac{y}{s(t)}\right) \quad \text{as} \quad t \to \infty, \tag{1.1}$$

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where $s(t)$ denotes the mean particle size at time $t > 0$ and ξ is a non-negative function in $L^1(\mathbb{R}_+, ydy)$ (see the extensive discussion in [4, 18] for Smoluchowski's coagulation equation [23, 24]). However, the validity of the dynamical scaling hypothesis (1.1) has up to now mainly be checked by numerical simulations [7, 13, 17] and is not yet established rigorously (except in a few particular cases [2, 12, 15, 16, 19]). The first difficulty encountered is actually the existence of the profile ξ and such a result has only been obtained recently for Smoluchowski's coagulation equation [6, 8]. The aim of the present paper is to investigate the existence of self-similar solutions for another coagulation equation which has been proposed in astrophysics by Oort & van de Hulst [20] and Safronov [21] in order to describe the aggregation of stellar objects. More precisely, the Oort-Hulst-Safronov (OHS) equation reads

$$\begin{aligned} \partial_t g &= Q_{OHS}(g)\,, \quad (t,y) \in (0,\infty)^2\,, && (1.2)\\ Q_{OHS}(g)(y) &:= -\partial_y \left(\int_0^y y_*\ a(y,y_*)\ g(y_*)\ dy_*\ g(y) \right) \\ &\quad - \int_y^\infty a(y,y_*)\ g(y_*)\ dy_*\ g(y)\,, && (1.3)\\ g(0) &= g_0\,, && (1.4) \end{aligned}$$

where the coagulation kernel a is a non-negative and symmetric function. Let us mention at this point that a solution to (1.2), (1.4) is expected to satisfy the conservation of mass

$$\int_0^\infty y\ g(t,y)\ dy = \int_0^\infty y\ g_0(y)\ dy \quad \text{for} \quad t \geq 0\,, \tag{1.5}$$

a property which holds true if a does not grow too fast for large values of y and y_*. For the coagulation kernels we consider in this paper, weak solutions constructed in [14] enjoy the mass conservation property.

For an homogeneous coagulation kernel satisfying

$$a(uy, uy_*) = u^\lambda a(y,y_*), \qquad (u,y,y_*) \in (0,\infty)^3,$$

for some $\lambda \in (-\infty, 1)$, it is rather natural to expect that the self-similar function g_s is in fact a self-similar solution to (1.2). Inserting the self-similar ansatz (1.1) in (1.2), we obtain $s(t) := (w(1-\lambda)t)^{1/(1-\lambda)}$ and that ξ satisfies

$$w\,[y\,\partial_y \xi(y) + 2\,\xi(y)] + Q_{OHS}(\xi)(y) = 0, \qquad y \in (0,\infty), \tag{1.6}$$

for some positive constant w. In addition, in view of (1.5), we require that $\xi \in L^1(0,\infty; y\,dy)$ with

$$\int_0^\infty \xi(y)\, y\, dy = \varrho, \tag{1.7}$$

for some positive constant ϱ. Observing that, if ξ is a solution to (1.6), (1.7) for the parameters $(1/(1-\lambda), \varrho)$, then the function $\xi_{a,b}$ defined by $\xi_{a,b}(y) = a\, \xi(b\, y)$ with

$$a = (w\,(1-\lambda))^{2/(1-\lambda)}, \qquad b = (w\,(1-\lambda))^{1/(1-\lambda)},$$

is a solution to (1.6), (1.7) for (w, ϱ), we conclude that we may assume that $w = \gamma := 1/(1-\lambda)$ without loss of generality.

From now on, we assume further that the coagulation kernel is given by

$$a(y, y_*) = y^\lambda + y_*^\lambda\,, \quad (y, y_*) \in (0,\infty)^2\,, \tag{1.8}$$

for some $\lambda \in (0,1)$ and prove the following result.

Theorem 1.1 *Given $\varrho \in (0,\infty)$ there exists a non-negative function $\xi \in L^1(0,\infty; (y + y^\lambda)\; dy)$, $y_0 \in (0,\infty)$ and $q \in (0,\infty)$ such that*

$$\int_0^\infty y\ \xi(y)\ dy = \varrho\,, \quad M_\lambda := \int_0^\infty y^\lambda\ \xi(y)\ dy \in (1,\gamma)\,, \tag{1.9}$$

$$\xi \in \mathcal{C}((0,\infty)\setminus\{y_0\}) \quad \textit{with} \quad \textit{Supp}\ \xi = [0, y_0] \quad \textit{and} \quad \xi(y_0-) > 0\,, \tag{1.10}$$

$$\lim_{y\to 0}\ y^\tau\ \xi(y) = q \quad \textit{with} \quad \tau := 2 - \frac{M_\lambda}{\gamma} \in (1, 1+\lambda) \quad \textit{and} \quad \gamma = \frac{1}{1-\lambda}\,, \tag{1.11}$$

and satisfying

$$\left(\gamma\ y - \int_0^y \left(y^\lambda + y_*^\lambda\right)\ y_*\ \xi(y_*)\ dy_*\right)\ \xi(y) = \left(\gamma - \int_y^\infty y_*^\lambda\ \xi(y_*)\ dy_*\right)\ \int_y^\infty \xi(y_*)\ dy_* \tag{1.12}$$

for $y \in (0, y_0)$. In addition, $g_s(t,y) := t^{-2\gamma}\ \xi\left(yt^{-\gamma}\right)$, $(t,y) \in (0,\infty)^2$, is a weak solution to (1.2), that is,

$$\frac{d}{dt}\int_0^\infty g_s(t,y)\ \vartheta(y)\ dy = \int_0^\infty \int_0^y \left(y_*\ \partial_y\vartheta(y) - \vartheta(y_*)\right)\ \left(y^\lambda + y_*^\lambda\right)\ g_s(t,y_*)\ g_s(t,y)\ dy_* dy$$

for every $t \in (0,\infty)$ and $\vartheta \in \mathcal{C}_0^\infty((0,\infty))$.

We first note that (1.12) is in fact a weak formulation of (1.6). Next, on the one hand, we point out that the profile ξ has a singularity at $y = 0$ and is actually not integrable near $y = 0$ by (1.11): a similar property is enjoyed by the profile of self-similar solutions to Smoluchowski's coagulation equation (see [4, 18] for formal arguments and [5, 9] for a rigorous proof). On the other hand, a striking difference between the profiles of self-similar solutions to the OHS and Smoluchowski coagulation equations is that the former are compactly supported with a discontinuity at the edge of the support while the latter belong to $\mathcal{C}^1((0,\infty))$ with infinite support [9]. Such a difference had already been noticed for the constant coagulation kernel ($a \equiv 1$), for which self-similar profiles to Smoluchowski's coagulation equation are explicitly given by $y \mapsto (4/\varrho)\, e^{-2y/\varrho}$ [4, 18] while self-similar profiles to (1.2) are also explicit and given by $y \mapsto (2/\varrho)\mathbf{1}_{[0,\varrho]}(y)$ [15].

To prove Theorem 1.1, we use a dynamical approach as in [6, 8, 10] relying on the fact that finding profiles ξ that satisfy (1.6), (1.7) amounts to find steady states to

$$\partial_t f = \gamma\ \left(y\ \partial_y f + 2\ f\right) + Q_{OHS}(f)\,, \quad (t,y)\in\mathbb{R}_+^2\,. \tag{1.13}$$

To this aim, we recall the following result.

Theorem 1.2 *Let X be a locally convex topological vector space and K be a non-empty compact and convex subset of X. If $\mathcal{F}\ : [0,\infty)\times K \to X$ is a semi-flow on X for which $\mathcal{F}(t,K)\subset K$ for each $t\ge 0$, then there is $x_0\in K$ such that $\mathcal{F}(t,x_0)=x_0$ for each $t\ge 0$. In other words, x_0 is a steady state for the semi-flow $\mathcal{F}$ in K.*

The proof of Theorem 1.2 relies on the Tychonov-Schauder fixed point theorem (or Brouwer fixed point theorem if $\dim X<\infty$) [1, 6, 10].

Applying Theorem 1.2 thus requires to find a functional setting in which (1.13) is well-posed, and a compact and convex invariant set as well. Existence of mass-conserving weak solutions to (1.2), (1.4) in $L^1(\mathbb{R}_+,(1+y)\ dy)$ has been established in [14]: noting that, if f is a solution to (1.13), (1.4), then

$$g(t,y) = \frac{1}{(1+t)^{2\gamma}}\ f\left(\ln(1+t), \frac{y}{(1+t)^\gamma}\right)$$

is a solution to (1.2), (1.4), we thus also obtain weak solutions to (1.13), (1.4) in the same functional setting. However, as far as uniqueness and continuous dependence are concerned, it turns out that it is more convenient to construct a semi-flow in $L^1(\mathbb{R}_+)$ for the cumulative distribution function F defined by

$$F(t,y) = \int_y^\infty f(t,y_*)\, dy_*, \qquad (t,y)\in\mathbb{R}_+^2\,.$$

Integrating (1.13) with respect to the size variable, we find that F solves

$$\begin{aligned}\partial_t F(t,y) + \gamma\partial_y(y\, F(t,y)) + y^{\lambda+1}\partial_y(F(t,y)^2) &= y^\lambda F(t,y)^2 + y^\lambda\, \partial_y F(t,y)\int_0^y F(t,y_*)\, dy_*\\ &+ (1+\lambda)\,\partial_y F(t,y)\int_0^y y_*^\lambda\, F(t,y_*)\, dy_* + \lambda F(t,y)\int_y^\infty y_*^{\lambda-1}\, F(t,y_*)\, dy_*.\end{aligned} \tag{1.14}$$

Thanks to Theorem 1.2, it is possible to prove that there exists a non-negative and non-increasing stationary solution $\Xi\in L^1(0,\infty;(1+y^{\lambda-1}+y)\, dy)$ to (1.14). In order to obtain a solution ξ to (1.6), (1.7), we need to show the differentiability of Ξ which we have yet been unable to prove. Actually, it turns out that (1.14) has a one-parameter family of non-smooth stationary weak solutions $y\mapsto \gamma\ r^{-\lambda}\ \mathbf{1}_{[0,r]}(y)$, $r\in\mathbb{R}_+$, and it is not clear whether the stationary solution Ξ to (1.14) we obtain does not belong to this family. To remedy for this drawback, we work with the inverse function $\Phi(t,.)$ of $F(t,.)$. Formally, Φ is a solution to

$$\begin{aligned}\partial_t\Phi(t,x) &= -\gamma\ (x\ \partial_x\Phi(t,x)+\Phi(t,x))+x\ \left(\int_0^x \Phi(t,x_*)^\lambda\ dx_*\right)\ \partial_x\Phi(t,x)\\ &+ \int_x^\infty \left(\Phi(t,x)^\lambda+\Phi(t,x_*)^\lambda\right)\ \Phi(t,x_*)\ dx_*\,. \end{aligned} \tag{1.15}$$

A natural functional setting for the well-posedness of (1.15) is the set

$$K_R := \left\{ \begin{array}{l} U \in L^1(0,\infty) \text{ is a non-negative and non-increasing function} \\ \text{such that } \|U\|_1 = \varrho,\ \|U\|_\infty \le R \text{ and } \|U^\lambda\|_1 \le R \end{array} \right\},$$

which is also invariant for sufficiently large R. Unfortunately, K_R is not convex since $\lambda \in (0,1)$. Therefore we cannot use Theorem 1.2 and this leads us to introduce the following modified equation:

$$\begin{aligned}\partial_t\Phi(t,x) &= -\gamma\ (x\ \partial_x\Phi(t,x)+\Phi(t,x))+x\ \left(\int_0^x \Phi(t,x_*)^\lambda\ dx_* + \delta x\right)\ \partial_x\Phi(t,x)\\ &+ \int_x^\infty \left(\Phi(t,x)^\lambda+\Phi(t,x_*)^\lambda+2\delta\right)\ \Phi(t,x_*)\ dx_*\,, \end{aligned} \tag{1.16}$$

where $\delta \in (0,1)$. Let us mention at this point that (1.16) can be obtained from (1.13) with the coagulation kernel $a_\delta(y,y_*) = y^\lambda + y_*^\lambda + 2\delta$ by the same procedure as (1.15). We then investigate the existence of steady states to (1.16). For that purpose, we first study in Section 2 the well-posedness of (1.16) supplemented with the initial datum

$$\Phi(0,x) = \Phi_0(x). \tag{1.17}$$

We then determine in Section 3 a compact and convex set which is left invariant by the semi-flow induced by (1.16), (1.17). The existence of a stationary solution to (1.16) then follows by applying Theorem 1.2. We obtain a stationary solution to (1.15) by letting $\delta \to 0$. Section 4 is then devoted to the analysis of the smoothness of this stationary solution. The proof of Theorem 1.1 is carried out in Section 5.

We finally introduce some notations: for any $u, v \in \mathbb{R}$, we define

$$u\wedge v = \min\{u,v\}, \qquad u\vee v = \max\{u,v\}, \qquad u_+ = \max\{u,0\}, \qquad \mathrm{sign}_+(u) = \mathrm{sign}(u_+).$$

For any $p \in [1,\infty]$ and $\zeta \in L^p(0,\infty)$, we set

$$\|\zeta\|_p = \|\zeta\|_{L^p(0,\infty)}.$$

2 Well-posedness of (1.16), (1.17)

In this section we prove the following theorem.

Theorem 2.1 *Assume that*

$$\begin{cases} \Phi_0 \in L^\infty(\mathbb{R}_+) \text{ is a non-negative and non-increasing compactly sup-} \\ \text{ported function such that Supp } \Phi_0 \subset [0, R_0] \text{ for some } R_0 > 0. \end{cases} \tag{2.1}$$

Then there is a unique function $\Phi \in \mathcal{C}([0,\infty); L^1(0,\infty))$ such that

- $\Phi(t,.)$ *is non-negative and non-increasing with compact support in* $[0, e^{\gamma t} R_0]$,
- $\|\Phi(t)\|_1 = \|\Phi_0\|_1$ *and* $\Phi \in L^\infty((0,t)\times(0,\infty))$,

and Φ satisfies (1.16) in the following weak sense

$$\begin{aligned} & \frac{d}{dt}\int_0^\infty \Phi(t,y)\ \vartheta(y)\ dy \\ = & \int_0^\infty \partial_y\vartheta(y)\ \left(\gamma\ y - \delta\ y^2 - y\int_0^y \Phi(t,y_*)^\lambda\ dy_*\right)\ \Phi(t,y)\ dy \\ - & \int_0^\infty \partial_y\vartheta(y)\ \int_y^\infty\int_0^y \left(\Phi(t,y_*)^\lambda + \Phi(t,y')^\lambda + 2\delta\right)\ \Phi(t,y_*)\ dy'dy_*\ dy \end{aligned} \tag{2.2}$$

for every $t \geq 0$ and $\vartheta \in \mathcal{C}_0^\infty((0,\infty))$.

Before proving Theorem 2.1, we first observe that, if φ is a weak solution to

$$\begin{aligned} \partial_t\varphi(t,x) &= x\ \left(\int_0^x \varphi(t,x_*)^\lambda\ dx_* + \delta x(1+t)^{\lambda\gamma}\right)\ \partial_x\varphi(t,x) \\ &+ \int_x^\infty \left(\varphi(t,x)^\lambda + \varphi(t,x_*)^\lambda + 2\delta(1+t)^{\lambda\gamma}\right)\ \varphi(t,x_*)\ dx_*\,, \end{aligned} \tag{2.3}$$

$$\varphi(0,x) = \Phi_0(x), \tag{2.4}$$

then the function Φ defined by

$$\Phi(t,x) = e^{-\gamma t}\varphi(e^t - 1, e^{-\gamma t}x), \qquad (t,x) \in (0,\infty)^2, \tag{2.5}$$

is a weak solution to (1.16), (1.17).

We now consider the existence part of Theorem 2.1 and show that there exists a weak solution to (2.3), (2.4). It then implies the existence of a weak solution to (1.16), (1.17) by (2.5).

Remark 2.2 *Formally, if Φ is the solution to (1.16), (1.17) given by Theorem 2.1, then $\tilde{f}(t,.) = -\frac{d}{dy}\Phi(t,.)^{-1}$ is a solution to (1.13) with coagulation kernel $a_\delta(y,y_*) = y^\lambda + y_*^\lambda + 2\delta$, and vice versa. A rigorous justification of this fact does not seem to be obvious and prevents us from using [14] to prove the existence part of Theorem 2.1.*

2.1 The regularized problem

We first investigate a regularized problem and prove the existence of a solution by the method of characteristics. Let $\varepsilon > 0$, $R > 1$ and $\chi_R \in \mathcal{C}^1(\mathbb{R}_+)$ be such that

$$\chi_R(x) = \begin{cases} x & \text{if} \quad x \leq R+1, \\ 2R & \text{if} \quad x \geq 4R, \end{cases}$$

and

$$0 \leq \chi_R(x) \leq 2R, \qquad 0 \leq \chi_R'(x) \leq 1 \qquad \text{for} \quad x \geq 0.$$

We consider the following equation

$$\begin{aligned} \partial_t \varphi(t,x) &= x \left(\int_0^x \mathcal{R}_\varepsilon(\varphi(t,x_*))\, dx_* + \delta\,(1+t)^{\lambda\gamma} \chi_{R_0}(x) \right) \partial_x \varphi(t,x) \\ &+ \int_x^\infty \left(\mathcal{R}_\varepsilon(\varphi(t,x)) + \mathcal{R}_\varepsilon(\varphi(t,x_*)) + 2\,\delta\,(1+t)^{\lambda\gamma} \right) \varphi(t,x_*)\, dx_*\,, \qquad (2.6) \\ \varphi(0,x) &= \Phi_0(x)\,, \qquad (2.7) \end{aligned}$$

where the initial datum Φ_0 fulfils (2.1) (R_0 is given in (2.1)) and

$$\mathcal{R}_\varepsilon(z) := (\varepsilon + z)^\lambda - \varepsilon^\lambda, \qquad z \in \mathbb{R}_+$$

is a $\mathcal{C}^1$-smooth approximation of $z \mapsto z^\lambda$. In particular, $\mathcal{R}_\varepsilon$ enjoys the following properties:

$$0 \leq \mathcal{R}_\varepsilon(z) \leq \left(\lambda\, \varepsilon^{\lambda-1}\, z \right) \wedge z^\lambda, \qquad |\mathcal{R}_\varepsilon(z) - \mathcal{R}_\varepsilon(z_*)| \leq \lambda\, \varepsilon^{\lambda-1} |z - z_*|, \quad (z, z_*) \in \mathbb{R}_+^2\,. \quad (2.8)$$

We now establish the existence of a solution to (2.6), (2.7) by a fixed point method. Let M, C_1, L_1 and T_ε be four positive real numbers, the values of which we will specify later. We denote by $\mathcal{H}_\varepsilon$ the set of non-negative functions $h \in \mathcal{C}([0,T_\varepsilon]; L^1(0,\infty))$ such that, for every $t \in [0, T_\varepsilon]$,

- $h(t,.)$ is non-increasing with compact support $\text{Supp}\, h(t,.) \subset [0, R_0]$,
- $\|h(t)\|_\infty \leq M$, $\|h(t)\|_1 \leq C_1$ and $|h(t,x) - h(t,x_*)| \leq L_1 |x - x_*|$ for every $(x, x_*) \in \mathbb{R}_+^2$.

For $h \in \mathcal{H}_\varepsilon$, we consider the following transport equation

$$\partial_t \varphi(t,x) + A_h(t,x)\, \partial_x \varphi(t,x) = B_h(t,x), \qquad (2.9)$$

where

$$\begin{aligned} A_h(t,x) &= -x \int_0^x \mathcal{R}_\varepsilon(h(t,x_*))\, dx_* - \delta\, x\, (1+t)^{\lambda\gamma} \chi_{R_0}(x), \\ B_h(t,x) &= \int_x^\infty \left(\mathcal{R}_\varepsilon(h(t,x)) + \mathcal{R}_\varepsilon(h(t,x_*)) + 2\,\delta\,(1+t)^{\lambda\gamma} \right) h(t,x_*)\, dx_*. \end{aligned}$$

Since $h(t,.)$ is non-increasing for every $t \in [0,T_\varepsilon]$, we have

$$x\,h(t,x) \le \int_0^x h(t,x_*)\,dx_*, \qquad (t,x) \in [0,T_\varepsilon] \times (0,\infty). \tag{2.10}$$

Owing to (2.8) and (2.10), A_h and B_h enjoy the following properties.

Lemma 2.3 *For $h \in \mathcal{H}_\varepsilon$, A_h is continuous on $[0,T_\varepsilon]\times[0,\infty)$ and, for every $(t,x) \in [0,T_\varepsilon]\times[0,\infty)$, we have*

$$\begin{aligned} -x\,\lambda\,\varepsilon^{\lambda-1}\,C_1 - 2\,\delta\,R_0\,x\,(1+T_\varepsilon)^{\lambda\gamma} \le &\quad A_h(t,x) \quad \le 0, \\ -2\,\lambda\varepsilon^{\lambda-1}C_1 - 6\,\delta\,R_0\,(1+T_\varepsilon)^{\lambda\gamma} \le &\quad \partial_x A_h(t,x) \quad \le 0. \end{aligned}$$

Moreover, $B_h(t,.)$ is a non-increasing function with compact support in $[0,R_0]$ for every $t \in [0,T_\varepsilon]$ and,

$$0 \ \le\ B_h(t,x) \le 2\,\left(M^\lambda + \delta\,(1+T_\varepsilon)^{\lambda\gamma}\right)\,C_1, \qquad (t,x) \in [0,T_\varepsilon]\times[0,\infty).$$

We then construct the characteristic curves associated to (2.9): for $h \in \mathcal{H}_\varepsilon$, $t \in [0,T_\varepsilon]$ and $x \in [0,\infty)$, it follows from the continuity of A_h and the boundedness of $\partial_x A_h$ established in Lemma 2.3 and the Cauchy-Lipschitz theorem that the ordinary differential equation

$$\frac{dX}{ds}(s;t,x) = A_h(s,X(s;t,x)), \tag{2.11}$$

$$X(t;t,x) = x, \tag{2.12}$$

has a unique global solution $X(.;t,x) \in \mathcal{C}^1([0,T_\varepsilon];\mathbb{R})$. Furthermore, Lemma 2.3 warrants that

$$x\,e^{-(\lambda\varepsilon^{\lambda-1}C_1+2\,\delta\,R_0\,(1+T_\varepsilon)^{\lambda\gamma})(s-t)} \le X(s;t,x) \le x \quad \text{for} \quad s \in [t,T_\varepsilon], \tag{2.13}$$

$$x \le X(s;t,x) \le x\,e^{(\lambda\varepsilon^{\lambda-1}C_1+2\,\delta\,R_0\,(1+T_\varepsilon)^{\lambda\gamma})(t-s)} \quad \text{for} \quad s \in [0,t]. \tag{2.14}$$

Proposition 2.4 *Consider $\Phi_0 \in W^{1,\infty}(\mathbb{R}_+)$ satisfying (2.1) and $h \in \mathcal{H}_\varepsilon$. Setting*

$$\varphi(t,x) = \Phi_0(X(0;t,x)) + \int_0^t B_h(s,X(s;t,x))\,ds, \qquad (t,x) \in [0,T_\varepsilon]\times[0,\infty), \tag{2.15}$$

then φ is the unique weak solution to (2.9) with initial datum Φ_0. In addition, $\varphi(t,.)$ is non-increasing and Supp $\varphi(t,.) \subset [0,R_0]$ for each $t \in [0,T_\varepsilon]$. Moreover, if $\varepsilon \in (0,\lambda^\gamma)$, $M = 1 + 2\|\Phi_0\|_\infty$, $C_1 = 2\|\Phi_0\|_1$, $L_1 = 4(\|\partial_x\Phi_0\|_\infty + M + (M^2/C_1))$ and

$$T_\varepsilon = \min\left(2^{1/(\lambda\gamma)} - 1\,,\ \frac{1}{8\,\lambda\,\varepsilon^{\lambda-1}\,\|\Phi_0\|_1 + 24\,\delta\,R_0}\right),$$

then $\varphi \in \mathcal{H}_\varepsilon$.

Proof. The first assertion of Proposition 2.4 is classical and the compactness of the support of $\varphi(t,.)$ follows from that of Φ_0 and B_h (see Lemma 2.3). We next investigate the behaviour of the L^∞- and L^1-norms of φ. We deduce from Lemma 2.3 that

$$0 \le \varphi(t,x) \le \|\Phi_0\|_\infty + 2\left(M^\lambda + \delta\,(1+T_\varepsilon)^{\lambda\gamma}\right) C_1\; T_\varepsilon \tag{2.16}$$

for every $(t,x) \in [0,T_\varepsilon]\times[0,\infty)$. Next, the change of variables $x_* = X(s;t,x)$ is a diffeomorphism for every $(s,t)\in[0,T_\varepsilon]^2$ and we have $x = X(t;s,x_*)$ with

$$\partial_x X(t;s,x_*) = \exp\left(\int_s^t \partial_x A_h(\sigma, X(\sigma;s,x_*))\,d\sigma\right). \tag{2.17}$$

Consequently, we deduce from (2.13), (2.14) and (2.15) that

$$\begin{aligned}\int_0^\infty \varphi(t,x)\,dx &= \int_0^\infty \Phi_0(x_*)\exp\left(\int_0^t \partial_x A_h(\sigma, X(\sigma;0,x_*))\,d\sigma\right)dx_*\\ &+ \int_0^t\int_0^\infty B_h(s,x_*)\exp\left(\int_s^t \partial_x A_h(\sigma, X(\sigma;s,x_*))\,d\sigma\right)dx_*\,ds.\end{aligned}$$

The non-positivity of $\partial_x A_h$ (see Lemma 2.3) implies that

$$\int_0^\infty \varphi(t,x)\,dx \le \int_0^\infty \Phi_0(x_*)\,dx_* + \int_0^t\int_0^\infty B_h(s,x_*)\,dx_*\,ds.$$

But, by (2.8), the monotonicity and the compactness of the support of $h(s,.)$ for $s\in[0,T_\varepsilon]$, we get

$$\begin{aligned}\int_0^t\int_0^\infty B_h(s,x_*)\,dx_*\,ds &\le 2\,\lambda\,\varepsilon^{\lambda-1}\int_0^t\int_0^\infty h(s,x_*)\int_{x_*}^\infty h(s,x)\,dx\,dx_*\,ds\\ &+ 2\,\delta(1+T_\varepsilon)^{\lambda\gamma}\int_0^t\int_0^{R_0}\int_{x_*}^{R_0} h(s,x)\,dx\,dx_*\,ds\\ &\le \lambda\,\varepsilon^{\lambda-1}\,T_\varepsilon\,C_1^2 + 2\,\delta\,R_0\,C_1\,T_\varepsilon\,(1+T_\varepsilon)^{\lambda\gamma}.\end{aligned}$$

Thus,

$$\|\varphi(t)\|_1 \le \|\Phi_0\|_1 + 2\,C_1\,T_\varepsilon\left(\lambda\,\varepsilon^{\lambda-1}\,C_1 + \delta\,R_0\,(1+T_\varepsilon)^{\lambda\gamma}\right), \qquad t\in[0,T_\varepsilon]. \tag{2.18}$$

We next turn to the Lipschitz property and put $L_0 := \|\partial_x\Phi_0\|_\infty$. For every $(x,x_*)\in(0,\infty)^2$, we have

$$\begin{aligned}|\varphi(t,x)-\varphi(t,x_*)| &\le |\Phi_0(X(0;t,x)) - \Phi_0(X(0;t,x_*))|\\ &+ \int_0^t |B_h(s,X(s;t,x)) - B_h(s,X(s;t,x_*))|\,ds.\end{aligned}$$

But, by (2.8),

$$\begin{aligned} & |B_h(s,X(s;t,x)) - B_h(s,X(s;t,x_*))| \\ \leq\ & 2\,M\left(\lambda\,\varepsilon^{\lambda-1}M + \delta\,(1+T_\varepsilon)^{\lambda\gamma}\right)|X(s;t,x) - X(s;t,x_*)| \\ +\ & \lambda\,\varepsilon^{\lambda-1}\,C_1\,|h(s,X(s;t,x)) - h(s,X(s;t,x_*))|. \end{aligned}$$

Consequently,

$$\begin{aligned} |\varphi(t,x) - \varphi(t,x_*)| &\leq L_0\,|X(0;t,x) - X(0;t,x_*)| \\ &+ \left(\lambda\,\varepsilon^{\lambda-1}(L_1\,C_1 + 2M^2) + 2\,\delta\,M\,(1+T_\varepsilon)^{\lambda\gamma}\right)\int_0^t |X(s;t,x) - X(s;t,x_*)|\,ds. \end{aligned}$$

We next deduce from (2.17) and Lemma 2.3 that, for $s \in [0,t]$,

$$|X(s;t,x) - X(s;t,x_*)| \leq \exp\Big(\left(2\,\lambda\,\varepsilon^{\lambda-1}C_1 + 6\,\delta\,R_0\,(1+T_\varepsilon)^{\lambda\gamma}\right)(t-s)\Big)\,|x - x_*|.$$

Consequently,

$$\begin{aligned} |\varphi(t,x) - \varphi(t,x_*)| \leq\ & |x - x_*|\Bigg[\left(L_0 + \frac{M^2}{C_1} + \frac{M}{3\,R_0}\right)\exp\left\{(2\lambda\,\varepsilon^{\lambda-1}\,C_1 + 6\,\delta\,R_0\,(1+T_\varepsilon)^{\lambda\gamma})\,T_\varepsilon\right\} \\ & +\ \frac{L_1}{2}\left(\exp\left\{(2\lambda\,\varepsilon^{\lambda-1}\,C_1 + 6\,\delta\,R_0\,(1+T_\varepsilon)^{\lambda\gamma})\,T_\varepsilon\right\} - 1\right)\Bigg]. \end{aligned} \tag{2.19}$$

Moreover, (2.17) implies that the function $x \mapsto X(s;t,x)$ is non-decreasing for every $(s,t) \in [0,T_\varepsilon]^2$. It thus follows from the monotonicity of Φ_0 and $B_h(s,.)$ (see Lemma 2.3) that $x \mapsto \varphi(t,x)$ is a non-increasing function.

Finally, if $\varepsilon \in (0,\lambda^\gamma)$, we put $M = 1 + 2\|\Phi_0\|_\infty$, $C_1 = 2\|\Phi_0\|_1$,

$$L_1 = 4\left(L_0 + \frac{M^2}{C_1} + M\right) \qquad \text{and} \qquad T_\varepsilon = \min\left(2^{1/(\lambda\gamma)} - 1\,,\ \frac{1}{4\,\lambda\,\varepsilon^{\lambda-1}\,C_1 + 24\,\delta\,R_0}\right),$$

and notice that $C_1 \leq R_0 M$. It then readily follows from (2.16), (2.18) and (2.19) that φ belongs to $\mathcal{H}_\varepsilon$. □

Theorem 2.5 *Consider $\varepsilon \in (0,\lambda^\gamma)$ and $\Phi_0 \in W^{1,\infty}(\mathbb{R}_+)$ satisfying (2.1). Then there exists a non-negative weak solution $\varphi^\varepsilon \in \mathcal{C}([0,\infty);L^1(0,\infty))$ to (2.6), (2.7) such that $\varphi^\varepsilon(t,.)$ is a non-increasing function for every $t \geq 0$ and φ^ε satisfies*

$$\mathit{Supp}\ \varphi^\varepsilon(t,.) \subset [0,R_0], \qquad \|\varphi^\varepsilon(t)\|_1 = \|\Phi_0\|_1, \qquad t \in (0,\infty).$$

To prove the existence of a weak solution to (2.6), (2.7), we need a preliminary lemma.

Lemma 2.6 *Consider $\varepsilon \in (0, \lambda^\gamma)$ and $\Phi_0 \in W^{1,\infty}(\mathbb{R}_+)$ satisfying (2.1). Assume that the parameters M, C_1, L_1 and T_ε are given by Proposition 2.4. Let h_1 and h_2 be two functions in $\mathcal{H}_\varepsilon$ and denote by X_1 and X_2 the associated characteristic curves. Setting $z := X_1 - X_2$, we have*

$$|z(s;t,x)| \le \lambda\, \varepsilon^{\lambda-1}\, T_\varepsilon\, \|h_1 - h_2\|_{\mathcal{C}([0,T_\varepsilon];L^1(0,\infty))}\, X_i(s;t,x)\, \exp\left(2\, T_\varepsilon\, (\lambda\, \varepsilon^{\lambda-1}\, C_1 + 6\, \delta\, R_0)\right), \quad (2.20)$$

for $0 \le s \le t \le T_\varepsilon$, $x \in [0,\infty)$ and $i = 1, 2$.

Proof. By (2.11), for $s \le t$, we have

$$\begin{aligned} |z(s;t,x)| &\le \int_s^t |A_{h_1}(\sigma, X_1(\sigma;t,x)) - A_{h_1}(\sigma, X_2(\sigma;t,x))|\, d\sigma \\ &+ \int_s^t |(A_{h_1} - A_{h_2})(\sigma, X_2(\sigma;t,x))|\, d\sigma. \end{aligned}$$

Since $(1+T_\varepsilon)^{\lambda\gamma} \le 2$, it follows from Lemma 2.3 that

$$\begin{aligned} |z(s;t,x)| &\le 2\,(\lambda\,\varepsilon^{\lambda-1}\, C_1 + 6\,\delta\, R_0) \int_s^t |z(\sigma;t,x)|\, d\sigma \\ &+ \int_s^t X_2(\sigma;t,x) \int_0^{X_2(\sigma;t,x)} |\mathcal{R}_\varepsilon(h_1(\sigma,x_*)) - \mathcal{R}_\varepsilon(h_2(\sigma,x_*))|\, dx_*\, d\sigma. \end{aligned}$$

For $s \le t$, (2.8) thus leads to

$$\begin{aligned} |z(s;t,x)| &\le 2\,(\lambda\,\varepsilon^{\lambda-1}\, C_1 + 6\,\delta\, R_0) \int_s^t |z(\sigma;t,x)|\, d\sigma \\ &+ \lambda\,\varepsilon^{\lambda-1}\, \|h_1 - h_2\|_{\mathcal{C}([0,T_\varepsilon];L^1(0,\infty))} \int_s^t X_2(\sigma;t,x)\, d\sigma. \end{aligned}$$

The Gronwall lemma and the monotonicity of $X_2(.;t,x)$ then imply that, for $s \le t$, (2.20) holds for $i = 2$. By symmetry of X_1 and X_2, we infer that (2.20) also holds for $i = 1$. $\square$

Proof of Theorem 2.5. Let M, C_1, L_1 and T_ε be the four parameters given by Proposition 2.4. We consider the map $\mathcal{T} : \mathcal{H}_\varepsilon \longrightarrow \mathcal{H}_\varepsilon$ defined by $\mathcal{T}(h) = \varphi$, where φ is given by (2.15). Let us check that $\mathcal{T}$ is continuous and compact for the topology of $\mathcal{C}([0,T_\varepsilon]; L^1(0,\infty))$.

Continuity of $\mathcal{T}$

Let $(h_n)_{n\ge1}$ be a sequence in $\mathcal{H}_\varepsilon$ that converges to $h \in \mathcal{H}_\varepsilon$. Let X_n and X be the characteristic curves defined by (2.11), (2.12) associated to A_{h_n} and A_h, respectively. We set $z_n = X_n - X$, $\varphi_n = \mathcal{T}(h_n)$ and $\varphi = \mathcal{T}(h)$ for $n \ge 1$. Then,

$$\begin{aligned} \|\varphi_n(t) - \varphi(t)\|_1 &\le \int_0^\infty |\Phi_0(X_n(0;t,x)) - \Phi_0(X(0;t,x))|\, dx \quad (=: J_1(t)) \\ &+ \int_0^\infty \int_0^t |B_{h_n}(s, X_n(s;t,x)) - B_h(s, X_n(s;t,x))|\, ds\, dx \quad (=: J_2(t)) \\ &+ \int_0^\infty \int_0^t |B_h(s, X_n(s;t,x)) - B_h(s, X(s;t,x))|\, ds\, dx. \quad (=: J_3(t)) \end{aligned}$$

Owing to the compactness of the support of Φ_0, we have

$$J_1(t) = \int_0^{R_0} |\Phi_0(X_n(0;t,x)) - \Phi_0(X(0;t,x))|\, dx \le \|\partial_x \Phi_0\|_\infty \int_0^{R_0} |z_n(0;t,x)|\, dx,$$

and we deduce from (2.14) and (2.20) that

$$J_1(t) \le \frac{1}{2}\, \lambda\, \varepsilon^{\lambda-1}\, T_\varepsilon\, \|\partial_x \Phi_0\|_\infty\, R_0^2\, \exp\big(T_\varepsilon\, (3\,\lambda\,\varepsilon^{\lambda-1}\, C_1 + 16\,\delta\, R_0)\big)\, \|h_n - h\|_{\mathcal{C}([0,T_\varepsilon];L^1(0,\infty))}. \tag{2.21}$$

Let us now consider $J_2(t)$. A change of variables yields

$$\begin{aligned} J_2(t) &= \int_0^t \int_0^\infty |B_{h_n}(s,x_*) - B_h(s,x_*)|\ \partial_x X_n(t;s,x_*)\ dx_* ds \\ &\le \int_0^t \int_0^\infty |B_{h_n}(s,x) - B_h(s,x)|\ dx ds\,, \end{aligned}$$

the last inequality being a consequence of the non-positivity of $\partial_x A_{h_n}$ and (2.17). Since both $h(s,.)$ and $h_n(s,.)$ are non-increasing and compactly supported for $s \in [0,T_\varepsilon]$, we obtain

$$\begin{aligned} J_2(t) &\le \int_0^t \int_0^\infty |\mathcal{R}_\varepsilon(h_n(s,x)) - \mathcal{R}_\varepsilon(h(s,x))| \int_x^\infty h_n(s,x_*)\, dx_*\, dx\, ds \\ &+ 2\int_0^t \int_0^{R_0} \big(\mathcal{R}_\varepsilon(h(s,x)) + \delta\,(1+s)^{\lambda\gamma}\big) \int_x^\infty |(h_n - h)(s,x_*)|\ dx_*\, dx\, ds \\ &+ \int_0^t \int_0^\infty h_n(s,x) \int_x^\infty |\mathcal{R}_\varepsilon(h_n(s,x_*)) - \mathcal{R}_\varepsilon(h(s,x_*))|\, dx_*\, dx\, ds. \end{aligned}$$

By (2.8), we get

$$\begin{aligned} J_2(t) &\le \lambda\,\varepsilon^{\lambda-1}\, C_1 \int_0^t\int_0^\infty |(h_n - h)(s,x)|\, dx\, ds + 2\,\delta\, R_0\, T_\varepsilon\, (1+T_\varepsilon)^{\lambda\gamma}\, \|h_n - h\|_{\mathcal{C}([0,T_\varepsilon];L^1(0,\infty))} \\ &+ \lambda\,\varepsilon^{\lambda-1}\, \|h_n - h\|_{\mathcal{C}([0,T_\varepsilon];L^1(0,\infty))} \int_0^t\int_0^{R_0} (2h(s,x) + h_n(s,x))\, dx\, ds, \end{aligned}$$

whence

$$J_2(t) \le 4\, T_\varepsilon \big(\lambda\,\varepsilon^{\lambda-1}\, C_1 + \delta\, R_0\big)\, \|h_n - h\|_{\mathcal{C}([0,T_\varepsilon];L^1(0,\infty))}. \tag{2.22}$$

It remains to handle $J_3(t)$. By the compactness of the support of $h(s,.)$ for $s \in [0,T_\varepsilon]$, we have

$$\begin{aligned} J_3(t) &\le \int_0^t \int_0^{R_0} |\mathcal{R}_\varepsilon(h(s,X_n(s;t,x))) - \mathcal{R}_\varepsilon(h(s,X(s;t,x)))| \int_{X_n(s;t,x)}^\infty h(s,x_*)\, dx_*\, dx\, ds \\ &+ \int_0^t \int_0^{R_0} \big(\mathcal{R}_\varepsilon(h(s,X(s;t,x))) + 2\,\delta\,(1+s)^{\lambda\gamma}\big) \left| \int_{X_n(s;t,x)}^{X(s;t,x)} h(s,x_*)\, dx_* \right| dx\, ds \\ &+ \int_0^t \int_0^{R_0} \left| \int_{X_n(s;t,x)}^{X(s;t,x)} h(s,x_*)\, \mathcal{R}_\varepsilon(h(s,x_*))\, dx_* \right| dx\, ds. \end{aligned}$$

Thanks to (2.8), we obtain

$$\begin{aligned} J_3(t) &\leq \lambda\,\varepsilon^{\lambda-1}\,C_1 \int_0^t \int_0^{R_0} |h(s,X_n(s;t,x)) - h(s,X(s;t,x))|\,dx\,ds \\ &+ 2\,M\left(\lambda\,\varepsilon^{\lambda-1}\,M + \delta\,(1+T_\varepsilon)^{\lambda\gamma}\right) \int_0^t \int_0^{R_0} |z_n(s;t,x)|\,dx\,ds. \end{aligned}$$

The Lipschitz continuity of h, (2.14) and (2.20) then imply that

$$\begin{aligned} J_3(t) \leq &\left(\lambda\,\varepsilon^{\lambda-1}\,M^2 + 2\,\delta\,M + \frac{1}{2}\,\lambda\,\varepsilon^{\lambda-1}\,C_1\,L_1\right)\lambda\,\varepsilon^{\lambda-1}\,T_\varepsilon^2\,R_0^2 \\ &\times \exp\left(T_\varepsilon\,(3\,\lambda\,\varepsilon^{\lambda-1}\,C_1 + 16\,\delta\,R_0)\right)\|h_n - h\|_{\mathcal{C}([0,T_\varepsilon];L^1(0,\infty))}. \end{aligned} \tag{2.23}$$

Finally, since

$$\|\varphi_n(t) - \varphi(t)\|_1 \leq J_1(t) + J_2(t) + J_3(t),$$

we infer from (2.21), (2.22) and (2.23) that $\mathcal{T}$ is continuous.

Compactness of $\mathcal{T}$

Let $(h_n)_{n\geq 1}$ be a sequence in $\mathcal{H}_\varepsilon$ and put $\varphi_n = \mathcal{T}(h_n)$ for $n \geq 1$. On the one hand, since φ_n belongs to $\mathcal{H}_\varepsilon$ for each $n \geq 1$, the sequence $(\varphi_n)_{n\geq 1}$ is bounded in $L^\infty(0,T_\varepsilon;W^{1,\infty}(0,R_0))$. On the other hand, we have $\partial_t\varphi_n = -A_{h_n}\,\partial_x\varphi_n + B_{h_n}$ by Lemma 2.3 and Proposition 2.4, from which we readily conclude that $(\partial_t\varphi_n)_{n\geq 1}$ is bounded in $L^\infty((0,T_\varepsilon)\times(0,R_0))$. By the Arzelà-Ascoli theorem, $(\varphi_n)_{n\geq 1}$ is then relatively compact in $\mathcal{C}([0,T_\varepsilon]\times[0,R_0])$, whence in $\mathcal{C}([0,T_\varepsilon];L^1(0,\infty))$ thanks to the compactness of the support of $\varphi_n(t,.)$ for each $t \in [0,T_\varepsilon]$ and $n \geq 1$.

Therefore, $\mathcal{H}_\varepsilon$ is a non-empty, convex, closed and bounded subset of $\mathcal{C}([0,T_\varepsilon];L^1(0,\infty))$, and $\mathcal{T}$ is a compact and continuous map from $\mathcal{H}_\varepsilon$ into $\mathcal{H}_\varepsilon$. The Schauder fixed point theorem ensures the existence of a fixed point of $\mathcal{T}$, that is a weak solution $\varphi^{\varepsilon,1} \in \mathcal{C}([0,T_\varepsilon];L^1(0,\infty))$ to (2.6), (2.7) such that $\varphi^{\varepsilon,1}(t,.)$ is non-negative and non-increasing with compact support in $[0,R_0]$ for each $t \in [0,T_\varepsilon]$.

Now, since Supp $\varphi^{\varepsilon,1}(t,.) \subset [0,R_0]$ for $t \in [0,T_\varepsilon]$ and $\chi_{R_0}(x) = x$ for $x \leq R_0 + 1$, we deduce from (2.6) that

$$\frac{d}{dt}\int_0^\infty \varphi^{\varepsilon,1}(t,x)\,dx = 0,$$

whence $\|\varphi^{\varepsilon,1}(t)\|_1 = \|\Phi_0\|_1$ for $t \in [0,T_\varepsilon]$. Observing that T_ε only depends on R_0 and $\|\Phi_0\|_1$, we may thus proceed as before with $\varphi^{\varepsilon,1}(T_\varepsilon)$ instead of Φ_0 and deduce the existence of a solution $\varphi^{\varepsilon,2} \in \mathcal{C}([0,T_\varepsilon];L^1(0,\infty))$ to (2.6) with initial condition $\varphi^{\varepsilon,1}(T_\varepsilon)$. Repeating this argument yields the existence of a solution $\varphi^\varepsilon \in \mathcal{C}([0,\infty);L^1(0,\infty))$ to (2.6), (2.7) that satisfies the desired properties. $\square$

The next task is to pass to the limit as $\varepsilon \to 0$. For that purpose, we need the following estimates.

Proposition 2.7 *Let $\varepsilon \in (0, \lambda^\gamma)$ and $\Phi_0 \in W^{1,\infty}(0,\infty)$ satisfy (2.1). The weak solution φ^ε to (2.6), (2.7) given by Theorem 2.5 is non-negative, $\varphi^\varepsilon(t,.)$ is non-increasing with compact support in $[0,R_0]$ and*

$$\begin{aligned} \|\varphi^\varepsilon(t)\|_1 &= \|\Phi_0\|_1, &(2.24)\\ \|\varphi^\varepsilon(t)\|_\infty &\leq (1+\|\Phi_0\|_\infty)\, e^{2\|\Phi_0\|_1 t\,(1+\delta\,(1+t)^{\lambda\gamma})}, &(2.25)\end{aligned}$$

for every $t \geq 0$.

Proof. All statements of Proposition 2.7 are actually a consequence of Theorem 2.5, except the L^∞-bound which we establish now. Let $p > 2$. Multiplying (2.6) by $p\,\varphi^\varepsilon(t,x)^{p-1}$ and recalling that Supp $\varphi^\varepsilon(t,.) \subset [0,R_0]$ for $t \geq 0$, we get

$$\begin{aligned} \frac{d}{dt}\int_0^\infty \varphi^\varepsilon(t,x)^p\,dx &= -\int_0^\infty \varphi^\varepsilon(t,x)^p \left(\int_0^x \mathcal{R}_\varepsilon(\varphi^\varepsilon(t,x_*))\,dx_* + \delta\,(1+t)^{\lambda\gamma} x\right)\,dx \\ &- \int_0^\infty \varphi^\varepsilon(t,x)^p \left(x\,\mathcal{R}_\varepsilon(\varphi^\varepsilon(t,x)) + \delta\,(1+t)^{\lambda\gamma}\,x\right)\,dx \\ &+ p\int_0^\infty \varphi^\varepsilon(t,x)\int_0^x \varphi^\varepsilon(t,x_*)^{p-1}\,\mathcal{R}_\varepsilon(\varphi^\varepsilon(t,x_*))\,dx_*\,dx \\ &+ p\int_0^\infty \left(\mathcal{R}_\varepsilon(\varphi^\varepsilon(t,x)) + 2\,\delta\,(1+t)^{\lambda\gamma}\right)\,\varphi^\varepsilon(t,x)\int_0^x \varphi^\varepsilon(t,x_*)^{p-1}\,dx_*\,dx. \end{aligned}$$

It then follows from (2.8), the non-negativity and the monotonicity of φ^ε that, for $t \in [0,T]$,

$$\begin{aligned} \frac{d}{dt}\int_0^\infty \varphi^\varepsilon(t,x)^p\,dx &\leq 2p\int_0^\infty \varphi^\varepsilon(t,x)\int_0^x \left(\varphi^\varepsilon(t,x_*)^{p+\lambda-1} + \delta\,(1+t)^{\lambda\gamma}\,\varphi^\varepsilon(t,x_*)^{p-1}\right)\,dx_*\,dx \\ &\leq 2p\,\|\Phi_0\|_1\left[\int_0^\infty \varphi^\varepsilon(t,x_*)^{p(p+\lambda-2)/(p-1)}\,\varphi^\varepsilon(t,x_*)^{(1-\lambda)/(p-1)}\,dx_*\right. \\ &+ \left.\delta\,(1+t)^{\lambda\gamma}\int_0^\infty \varphi^\varepsilon(t,x_*)^{p(p-2)/(p-1)}\,\varphi^\varepsilon(t,x_*)^{1/(p-1)}\,dx_*\right]. \end{aligned}$$

Since $p > 2$, the Young inequality implies that

$$\frac{d}{dt}\int_0^\infty \varphi^\varepsilon(t,x)^p\,dx \leq 2p\,\|\Phi_0\|_1(1+\delta\,(1+t)^{\lambda\gamma})\left[\int_0^\infty \varphi^\varepsilon(t,x)^p\,dx + \|\Phi_0\|_1\right].$$

Thus, we infer from the Gronwall lemma that

$$\|\varphi^\varepsilon(t)\|_p^p \leq (\|\Phi_0\|_p^p + \|\Phi_0\|_1)\,e^{2p\|\Phi_0\|_1 t\,(1+\delta\,(1+t)^{\lambda\gamma})}.$$

Hence,

$$\|\varphi^\varepsilon(t)\|_p \leq (\|\Phi_0\|_p + \|\Phi_0\|_1^{1/p})\,e^{2\|\Phi_0\|_1 t\,(1+\delta\,(1+t)^{\lambda\gamma})}.$$

Letting $p \longrightarrow \infty$ leads to (2.25). $\square$

2.2 Proof of Theorem 2.1

The existence part of Theorem 2.1 is a straightforward consequence of (2.5) and the following proposition.

Proposition 2.8 *Let $\Phi_0 \in L^\infty(0,\infty)$ satisfy (2.1). Then, there exists a weak solution $\varphi \in \mathcal{C}([0,\infty); L^1(0,\infty))$ to (2.3), (2.4) such that $\varphi(t,.)$ is non-negative and non-increasing with compact support in $[0,R_0]$,*

$$\|\varphi(t)\|_1 = \|\Phi_0\|_1 \qquad \text{and} \qquad \sup_{0\le s\le t} \|\varphi(s)\|_\infty < \infty,$$

for every $t \ge 0$.

Proof. We fix $k_0 \ge 1$ such that $k_0 > \lambda^{-\gamma}$ and $T>0$. Let $(\Phi_0^k)_{k\ge k_0}$ be a sequence of functions from $W^{1,\infty}(\mathbb{R}_+)$ such that Φ_0^k is non-negative and non-increasing with compact support in $[0,R_0]$, $\Phi_0^k(x) \le 2\Phi_0(x)$ a.e. and $(\Phi_0^k)_{k\ge k_0}$ converges towards Φ_0 in $L^1(\mathbb{R}_+)$.

For all $k \ge k_0$, we set $\varepsilon_k = 1/k < \lambda^\gamma$ and we denote by $\varphi^k = \varphi^{\varepsilon_k}$ the solution to (2.6) with initial condition Φ_0^k given by Theorem 2.5. It follows from (2.6) and Proposition 2.7 that

$$\begin{aligned}
(\varphi^k)_{k\ge k_0} \qquad & \text{is bounded in } L^\infty((0,T)\times(0,R_0)) \cap L^\infty(0,T;BV(0,R_0)),\\
(\partial_t\varphi^k)_{k\ge k_0} \qquad & \text{is bounded in } L^\infty(0,T;W^{1,1}(0,R_0)').
\end{aligned}$$

Since $L^\infty(0,R_0)\cap BV(0,R_0)$ is compactly embedded in $L^1(0,R_0)$ and $L^1(0,R_0)$ is continuously embedded in $W^{1,1}(0,R_0)'$, we infer from [22, Corollary 4] that $(\varphi^k)_{k\ge k_0}$ is relatively compact in $\mathcal{C}([0,T];L^1(0,R_0))$, whence in $\mathcal{C}([0,T];L^1(0,\infty))$ since φ^k identically vanishes in $(0,\infty)\times(R_0,\infty)$ for each $k\ge k_0$. Consequently, there exists $\varphi \in \mathcal{C}([0,T];L^1(0,\infty))$ such that, up to an extraction, (φ^k) converges towards φ in $\mathcal{C}([0,T];L^1(0,\infty))$ and a.e. on $[0,T]\times(0,\infty)$. Let $\vartheta \in \mathcal{C}_0^\infty((0,\infty))$. Then, recalling that Supp $\varphi^k(t,.) \subset [0,R_0]$ and $\chi_{R_0}(x) = x$ for $x \le R_0+1$, we deduce from (2.6) that, for every $t\in[0,T]$,

$$\begin{aligned}
&\int_0^\infty \varphi^k(t,x)\,\vartheta(x)\,dx - \int_0^\infty \Phi_0^k(x)\,\vartheta(x)\,dx\\
&= -\int_0^t \int_0^\infty \varphi^k(s,x)\,\partial_x\vartheta(x)\,x \left(\int_0^x \mathcal{R}_{\varepsilon_k}(\varphi^k(s,x_*))\,dx_* + \delta\,x\,(1+s)^{\lambda\gamma}\right) dx\,ds\\
&\quad -\int_0^t \int_0^\infty \varphi^k(s,x)\,\vartheta(x) \left(\int_0^x \mathcal{R}_{\varepsilon_k}(\varphi^k(s,x_*))\,dx_* + x\,\mathcal{R}_{\varepsilon_k}(\varphi^k(s,x)) + 2\,\delta\,x\,(1+s)^{\lambda\gamma}\right) dx\,ds\\
&\quad +\int_0^t \int_0^\infty \vartheta(x) \int_x^\infty \varphi^k(s,x_*)\,\left(\mathcal{R}_{\varepsilon_k}(\varphi^k(s,x)) + \mathcal{R}_{\varepsilon_k}(\varphi^k(s,x_*)) + 2\,\delta\,(1+s)^{\lambda\gamma}\right)\,dx_*\,dx\,ds.
\end{aligned}$$

We let $k\to\infty$ thanks to the dominated convergence theorem and get that φ is a weak solution to (2.3), (2.4). The properties satisfied by φ follow easily from Proposition 2.7. $\square$

The uniqueness assertion of Theorem 2.1 is actually a consequence of the following result:

Proposition 2.9 *Consider two functions Φ_0 and $\hat{\Phi}_0$ fulfilling the assumptions (2.1). If Φ and $\hat{\Phi}$ are weak solutions to (1.16), (1.17) with initial data Φ_0 and $\hat{\Phi}_0$, respectively, and $T>0$, there exists $C(T)$ depending only on λ, $\|\Phi_0\|_\infty$, $\|\hat{\Phi}_0\|_\infty$, R_0 and T such that*

$$\left\|\Phi(t)-\hat{\Phi}(t)\right\|_1 \le C(T)\ \left\|\Phi_0-\hat{\Phi}_0\right\|_1 \qquad \textit{for} \quad t\in[0,T]\,. \tag{2.26}$$

Proof. Let $T>0$. By Proposition 2.8 and (2.5), the support of $\Phi(t,.)$ and $\hat{\Phi}(t,.)$ is contained in $[0,R_0\ e^{\gamma t}]$ for each $t\in[0,T]$ and both Φ and $\hat{\Phi}$ belong to $L^\infty((0,T)\times(0,\infty))$. Consequently, we have

$$\Lambda := \sup_{t\in[0,T]} \left\{ \left\|\Phi(t)^\lambda\right\|_1 \vee \left\|\hat{\Phi}(t)^\lambda\right\|_1 \right\} \le R_0\ e^{\gamma T} \sup_{t\in[0,T]} \left\{ \|\Phi(t)\|_\infty^\lambda \vee \left\|\hat{\Phi}(t)\right\|_\infty^\lambda \right\} < \infty\,,$$

and notice that the monotonicity of Φ and $\hat{\Phi}$ imply that

$$x\ \Phi(t,x)^\lambda \le \int_0^x \Phi(t,x_*)^\lambda\ dx_* \le \Lambda \quad \text{and} \quad x\ \hat{\Phi}(t,x)^\lambda \le \int_0^x \hat{\Phi}(t,x_*)^\lambda\ dx_* \le \Lambda \tag{2.27}$$

for $x>0$. We put $E:=\Phi-\hat{\Phi}$ and $\sigma=\ \mathrm{sign}(E)$ and only give a formal proof of (2.26) below as both Φ and $\hat{\Phi}$ do not have the required smoothness to justify the forthcoming computations. Nevertheless, a rigorous proof can be performed by approximation arguments as in [3]. We infer from (1.16) that

$$\begin{aligned}
\frac{d}{dt}\|E(t)\|_1 &= -\gamma \int_0^\infty \left(x\ \partial_x|E(t,x)| + |E(t,x)|\right)\ dx \\
&+ \delta \int_0^\infty x^2\ \partial_x|E(t,x)|\ dx + 2\delta \int_0^\infty \sigma(t,x) \int_x^\infty E(t,x_*)\ dx_* dx \\
&+ \frac{1}{2} \int_0^\infty x \left(\int_0^x \left(\Phi^\lambda + \hat{\Phi}^\lambda\right)(t,x_*)\ dx_* \right) \partial_x|E(t,x)|\ dx \\
&+ \frac{1}{2} \int_0^\infty x\ \sigma(t,x) \left(\int_0^x \left(\Phi^\lambda - \hat{\Phi}^\lambda\right)(t,x_*)\ dx_* \right) \partial_x \left(\Phi+\hat{\Phi}\right)(t,x)\ dx \\
&+ \int_0^\infty \sigma(t,x) \int_x^\infty \left(\Phi(t,x)^\lambda\ \Phi(t,x_*) - \hat{\Phi}(t,x)^\lambda\ \hat{\Phi}(t,x_*)\right)\ dx_* dx \\
&+ \int_0^\infty \sigma(t,x) \int_x^\infty \left(\Phi^{1+\lambda} - \hat{\Phi}^{1+\lambda}\right)(t,x_*)\ dx_* dx
\end{aligned}$$

Integrating by parts the first, third and fifth terms of the right-hand side of the above equality and using the fact that $|\sigma(t,x)| \le 1$, we obtain

$$\begin{aligned}
\frac{d}{dt}\|E(t)\|_1 &\le \left[(-\gamma\ x + \delta\ x^2)\ |E(t,x)|\right]_0^\infty \\
&+ \frac{1}{2}\left[x\ \left(\int_0^x \left(\Phi(t,x_*)^\lambda + \hat{\Phi}(t,x_*)^\lambda\right)\ dx_*\right)\ |E(t,x)|\right]_0^\infty \\
&+ 2\delta\ \int_0^\infty \int_x^\infty |E(t,x_*)|\ dx_* dx - 2\delta\ \int_0^\infty x\ |E(t,x)|\ dx \quad (=: -I_1(t)) \\
&- \frac{1}{2}\int_0^\infty \int_0^x \left(\Phi^\lambda + \hat{\Phi}^\lambda\right)(t,x_*)\ dx_*\ |E(t,x)|\ dx \quad (=: -I_2(t)) \\
&- \frac{1}{2}\int_0^\infty x\ \left(\Phi^\lambda + \hat{\Phi}^\lambda\right)(t,x)\ |E(t,x)|\ dx \quad (=: -I_3(t)) \\
&+ \frac{1}{2}\int_0^\infty x\ \left(\int_0^x \left|\Phi^\lambda - \hat{\Phi}^\lambda\right|(t,x_*)\ dx_*\right)\ \left|\partial_x\left(\Phi + \hat{\Phi}\right)(t,x)\right|\ dx \quad (=: I_4(t)) \\
&+ \int_0^\infty \int_x^\infty \left|\Phi(t,x)^\lambda\ \Phi(t,x_*) - \hat{\Phi}(t,x)^\lambda\ \hat{\Phi}(t,x_*)\right|\ dx_* dx \quad (=: I_5(t)) \\
&+ \int_0^\infty \int_x^\infty \left|\Phi^{1+\lambda} - \hat{\Phi}^{1+\lambda}\right|(t,x_*)\ dx_* dx\,. \quad (=: I_6(t))
\end{aligned}$$

Owing to the compactness of the support of $\Phi(t,.)$ and $\hat{\Phi}(t,.)$, the boundary terms in the previous inequality vanish. Also, we clearly have $I_1(t) \ge 0$ by the Fubini theorem. Consequently,

$$\frac{d}{dt}\|E(t)\|_1 \le -I_2(t) - I_3(t) + I_4(t) + I_5(t) + I_6(t)\,. \tag{2.28}$$

Thanks to the monotonicity of $\Phi(t,.)$ and $\hat{\Phi}(t,.)$ with respect to x, $I_4(t)$ also reads

$$\begin{aligned}
2\ I_4(t) &= -\int_0^\infty x\ \left(\int_0^x \left|\Phi^\lambda - \hat{\Phi}^\lambda\right|(t,x_*)\ dx_*\right)\ \partial_x\left(\Phi + \hat{\Phi}\right)(t,x)\ dx \\
&= -\left[x\ \left(\int_0^x \left|\Phi^\lambda - \hat{\Phi}^\lambda\right|(t,x_*)\ dx_*\right)\ \left(\Phi + \hat{\Phi}\right)(t,x)\right]_0^\infty \\
&+ \int_0^\infty x\ \left|\Phi^\lambda - \hat{\Phi}^\lambda\right|(t,x)\ \left(\Phi + \hat{\Phi}\right)(t,x)\ dx \\
&+ \int_0^\infty \int_0^x \left|\Phi^\lambda - \hat{\Phi}^\lambda\right|(t,x_*)\ dx_*\ \left(\Phi + \hat{\Phi}\right)(t,x)\ dx\,.
\end{aligned}$$

As in the above computation, the boundary terms vanish. Since

$$(\Phi + \hat{\Phi})(t,x) = 2\ \left(\Phi \wedge \hat{\Phi}\right)(t,x) + |E(t,x)|\,, \tag{2.29}$$

we infer from the mean value theorem that

$$
\begin{aligned}
2\ I_4(t) &\leq 2\lambda \int_0^\infty x\ \left(\Phi \wedge \hat{\Phi}\right)^\lambda (t,x)\ |E(t,x)|\ dx \\
&+ \int_0^\infty x\ \left|\Phi^\lambda - \hat{\Phi}^\lambda\right| (t,x)\ |E(t,x)|\ dx \\
&+ 2\lambda \int_0^\infty \int_0^x \left(\Phi \wedge \hat{\Phi}\right)^{\lambda-1} (t,x_*)\ |E(t,x_*)|\ dx_*\ \left(\Phi \wedge \hat{\Phi}\right)(t,x)\ dx \\
&+ \int_0^\infty \int_0^x \left|\Phi^\lambda - \hat{\Phi}^\lambda\right| (t,x_*)\ dx_*\ |E(t,x)|\ dx\,.
\end{aligned}
$$

Using the monotonicity of $\Phi(t,.)$ and $\hat{\Phi}(t,.)$ with respect to x and (2.27), we further obtain

$$
\begin{aligned}
2\ I_4(t) &\leq 2\lambda\ \Lambda\ \|E(t)\|_1 + 2\ \Lambda\ \|E(t)\|_1 \\
&+ 2\lambda \int_0^\infty \int_0^x \left(\Phi \wedge \hat{\Phi}\right)^{\lambda-1} (t,x)\ |E(t,x_*)|\ dx_*\ \left(\Phi \wedge \hat{\Phi}\right)(t,x)\ dx \\
&+ 2\ \Lambda\ \|E(t)\|_1 \\
&\leq 2(\lambda+2)\ \Lambda\ \|E(t)\|_1 + 2\lambda \int_0^\infty \int_{x_*}^\infty \left(\Phi \wedge \hat{\Phi}\right)^\lambda (t,x)\ dx\ |E(t,x_*)|\ dx_* \\
I_4(t) &\leq 2(\lambda+1)\ \Lambda\ \|E(t)\|_1\,.
\end{aligned} \tag{2.30}
$$

Next, by (2.29), the Fubini theorem and the mean value theorem, we have

$$
\begin{aligned}
I_5(t) &\leq \frac{1}{2} \int_0^\infty \int_x^\infty \left(\Phi^\lambda + \hat{\Phi}^\lambda\right)(t,x)\ |E(t,x_*)|\ dx_* dx \\
&+ \frac{1}{2} \int_0^\infty \int_x^\infty \left(\Phi + \hat{\Phi}\right)(t,x_*)\ \left|\Phi^\lambda - \hat{\Phi}^\lambda\right| (t,x)\ dx_* dx \\
&\leq \frac{1}{2} \int_0^\infty \int_0^x \left(\Phi^\lambda + \hat{\Phi}^\lambda\right)(t,x_*)\ dx_*\ |E(t,x)|\ dx \\
&+ \lambda \int_0^\infty \int_x^\infty \left(\Phi \wedge \hat{\Phi}\right)(t,x_*)\ \left(\Phi \wedge \hat{\Phi}\right)^{\lambda-1} (t,x)\ |E(t,x)|\ dx_* dx \\
&+ \frac{1}{2} \int_0^\infty \int_x^\infty \left|\Phi^\lambda - \hat{\Phi}^\lambda\right| (t,x)\ |E(t,x_*)|\ dx_* dx \\
&\leq I_2(t) + \lambda \int_0^\infty \int_x^\infty \left(\Phi \wedge \hat{\Phi}\right)^\lambda (t,x_*)\ |E(t,x)|\ dx_* dx + \Lambda\ \|E(t)\|_1\,,
\end{aligned}
$$

the last inequality resulting from the monotonicity of $\Phi(t,.)$ and $\hat{\Phi}(t,.)$ with respect to x. We therefore end up with

$$
I_5(t) \leq I_2(t) + (\lambda+1)\ \Lambda\ \|E(t)\|_1\,. \tag{2.31}
$$

Next, using once more the monotonicity of $\Phi(t,.)$ and $\hat{\Phi}(t,.)$ with respect to x and the mean value theorem, we obtain

$$\begin{aligned} I_6(t) &\le (1+\lambda) \int_0^\infty \int_x^\infty \left(\Phi(t,x_*) \vee \hat{\Phi}(t,x_*)\right)^\lambda \ |E(t,x_*)| \ dx_* dx \\ &\le (1+\lambda) \int_0^\infty \left(\Phi(t,x) \vee \hat{\Phi}(t,x)\right)^\lambda \int_x^\infty |E(t,x_*)| \ dx_* dx \\ &\le 2 \ (1+\lambda) \ \Lambda \ \|E(t)\|_1 \,. \end{aligned} \tag{2.32}$$

Since I_3 is non-negative, we infer from (2.28), (2.30), (2.31) and (2.32) that

$$\frac{d}{dt}\|E(t)\|_1 \le 5 \ (1+\lambda) \ \Lambda \ \|E(t)\|_1$$

for $t \in [0,T]$, whence (2.26). □

3 Stationary solutions to (1.15)

Consider $\varrho \in (0,\infty)$. To establish the existence of a steady state Ψ to (1.15) satisfying $\|\Psi\|_1 = \varrho$, we proceed in two steps and first show that, for each $\delta \in (0,1)$, there is a stationary solution Ψ_δ to (1.16) such that $\|\Psi_\delta\|_1 = \varrho$. We next prove that the family $(\Psi_\delta)_{\delta\in(0,1)}$ belongs to a compact subset of $L^1(0,\infty)$ and that the cluster points of $(\Psi_\delta)_{\delta\in(0,1)}$ are stationary solutions to (1.15) satisfying the required L^1-constraint.

In order to apply Theorem 1.2 to the semi-flow associated to (1.16), (1.17), we have to identify a compact and convex subset of $L^1(0,\infty)$ which is left invariant by the semi-flow. We first recall that, when Φ_0 satisfies (2.1), then $\Phi(t,.)$ is compactly supported with $\mathrm{Supp}\ \Phi(t,.) \subset [0, R_0\ e^{\gamma t}]$ and

$$\int_0^\infty \Phi(t,x)\ dx = \varrho := \int_0^\infty \Phi_0(x)\ dx\,, \quad \text{for} \quad t \ge 0\,. \tag{3.1}$$

We next investigate the time evolution of the L^∞-norm.

Lemma 3.1 *Consider $\delta \in (0,1)$ and assume that Φ_0 satisfies (2.1). Denoting by Φ the corresponding solution to (1.16), (1.17), we have*

$$\|\Phi(t)\|_\infty \le m(t)\,, \quad t \ge 0 \tag{3.2}$$

where m is the solution to

$$\frac{dm}{dt}(t) = 2\left(m(t)^\lambda + \delta\right)\ \varrho\ - \gamma\ m(t)\,, \quad m(0) = \|\Phi_0\|_\infty\,, \tag{3.3}$$

the parameter ϱ being defined in (3.1).

Proof. We fix $T>0$ and recall that $\mathrm{Supp}\,\Phi(t,.)\subset[0,x_T]$ for $t\in[0,T]$ with $x_T := R_0\, e^{\gamma T}$. For $t\in[0,T]$, it follows from (1.16), (3.1) and the monotonicity and non-negativity of Φ that

$$\begin{aligned}
& \frac{d}{dt}\int_0^\infty (\Phi(t,x)-m(t))_+\ dx = \frac{d}{dt}\int_0^{x_T} (\Phi(t,x)-m(t))_+\ dx \\
\le\ & \gamma\int_0^{x_T} \mathrm{sign}_+(\Phi(t,x)-m(t))\ (\Phi(t,x)-m(t)-\Phi(t,x))\ dx \\
-\ & 2\delta\int_0^{x_T} x\ (\Phi(t,x)-m(t))_+\ dx \\
+\ & 2\delta\int_0^{x_T} \mathrm{sign}_+(\Phi(t,x)-m(t))\int_x^\infty \Phi(t,x_*)\ dx_* dx \\
+\ & \int_0^{x_T} \mathrm{sign}_+(\Phi(t,x)-m(t))\int_x^\infty \Phi(t,x_*)\ \left(\Phi(t,x)^\lambda+\Phi(t,x_*)^\lambda\right)\ dx_* dx \\
-\ & \int_0^{x_T} \mathrm{sign}_+(\Phi(t,x)-m(t))\ \frac{dm}{dt}(t)\ dx \\
\le\ & \int_0^{x_T} \mathrm{sign}_+(\Phi(t,x)-m(t))\ \left(2\delta\varrho-\gamma\ m(t)-\frac{dm}{dt}(t)\right)\ dx \\
+\ & 2\ \varrho\int_0^{x_T} \mathrm{sign}_+(\Phi(t,x)-m(t))\ \Phi(t,x)^\lambda\ dx\,.
\end{aligned}$$

By (3.3), we have the lower bound $m(t)\ge m(0)\ e^{-\gamma t}\ge m(0)\ e^{-\gamma T}$, and we obtain

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty (\Phi(t,x)-m(t))_+\ dx\ &\le\ 2\ \varrho\int_0^{x_T} \mathrm{sign}_+(\Phi(t,x)-m(t))\ \left(\Phi(t,x)^\lambda-m(t)^\lambda\right)\ dx \\
&\le\ 2\ \lambda\ \varrho\ m(t)^{\lambda-1}\int_0^{x_T}(\Phi(t,x)-m(t))_+\ dx \\
&\le\ 2\ \lambda\ \varrho\ m(0)^{\lambda-1}\ e^T\int_0^{x_T}(\Phi(t,x)-m(t))_+\ dx \\
&\le\ 2\ \lambda\ \varrho\ m(0)^{\lambda-1}\ e^T\int_0^{\infty}(\Phi(t,x)-m(t))_+\ dx\,.
\end{aligned}$$

Consequently,

$$\int_0^\infty (\Phi(t,x)-m(t))_+\ dx \le C(T)\int_0^\infty (\Phi_0(x)-m(0))_+\ dx = 0\,,$$

from which the inequality (3.2) readily follows for $t\in[0,T]$. Since T was arbitrarily chosen, we obtain the expected result. $\square$

Having excluded the occurrence of large values of Φ throughout time evolution, we next turn to a refined estimate on the propagation of the support of Φ.

Lemma 3.2 *Consider $\delta \in (0,1)$ and assume that Φ_0 satisfies (2.1). Denoting by Φ the corresponding solution to (1.16), (1.17), we have*

$$Supp\ \Phi(t,.) \subset [0, R(t)]\,, \quad t \geq 0 \tag{3.4}$$

where R is the solution to

$$\frac{dR}{dt}(t) = \gamma\ R(t) - \delta\ R(t)^2\,, \quad R(0) = R_0\,. \tag{3.5}$$

Proof. For $t \in (0,\infty)$, it follows from (1.16) and the Fubini theorem that

$$\begin{aligned}
\frac{d}{dt}\int_{R(t)}^\infty \Phi(t,x)\ dx &= -\frac{dR}{dt}(t)\ \Phi(t,R(t)) - \gamma\ \Big[x\ \Phi(t,x)\Big]_{R(t)}^\infty \\
&+ \left[x\ \Phi(t,x)\left(\delta\ x + \int_0^x \Phi(t,x_*)^\lambda\ dx_*\right)\right]_{R(t)}^\infty \\
&- \int_{R(t)}^\infty \left(\delta\ x + \int_0^x \Phi(t,x_*)^\lambda\ dx_*\right)\ \Phi(t,x)\ dx \\
&- \int_{R(t)}^\infty x\ \left(\delta + \Phi(t,x)^\lambda\right)\ \Phi(t,x)\ dx \\
&+ \int_{R(t)}^\infty \int_{R(t)}^x \left(\Phi(t,x)^\lambda + \Phi(t,x_*)^\lambda + 2\delta\right)\ \Phi(t,x)\ dx_* dx \\
&= \left(-\frac{dR}{dt}(t) + \gamma\ R(t) - \delta\ R(t)^2 - R(t)\ \int_0^{R(t)} \Phi(t,x_*)^\lambda\ dx_*\right)\Phi(t,R(t)) \\
&- \int_{R(t)}^\infty \int_0^x \left(2\delta + \Phi(t,x)^\lambda + \Phi(t,x_*)^\lambda\right)\ \Phi(t,x)\ dx_* dx \\
&+ \int_{R(t)}^\infty \int_{R(t)}^x \left(\Phi(t,x)^\lambda + \Phi(t,x_*)^\lambda + 2\delta\right)\ \Phi(t,x)\ dx_* dx \\
&\leq 0\,,
\end{aligned}$$

from which we deduce (3.4) by integration. $\square$

Observe that the estimate on the expansion on the support of Φ obtained in the previous lemma heavily depends on δ and will thus not be useful to pass to the limit as $\delta \to 0$. For that purpose, a control on the behaviour of Φ for large x which does not depend on δ is obtained in the next lemma.

Lemma 3.3 *Consider $\delta \in (0,1)$ and assume that Φ_0 satisfies (2.1). Denoting by Φ the corresponding solution to (1.16), (1.17), we have*

$$\mathcal{L}(t) := \int_0^\infty x^{(1-\lambda)/\lambda}\ \Phi(t,x)\ dx \leq \ell(t)\,, \quad t \geq 0\,, \tag{3.6}$$

where

$$\frac{d\ell}{dt}(t) = \frac{1}{\lambda}\ \ell(t) - \frac{1-\lambda}{(1+\lambda)\lambda^{1+\lambda}}\ \ell(t)^{1+\lambda}\,, \quad \ell(0) = \mathcal{L}(0) = \int_0^\infty x^{(1-\lambda)/\lambda}\ \Phi_0(x)\ dx\,.$$

Proof. For $t \in (0,\infty)$, it follows from (1.16), the compactness of the support of $\Phi(t,.)$ and the Fubini theorem that

$$\begin{aligned}
\frac{d\mathcal{L}}{dt}(t) &= \frac{\mathcal{L}(t)}{\lambda} - \frac{1}{\lambda} \int_0^\infty x^{(1-\lambda)/\lambda}\ \left(\delta\ x + \int_0^x \Phi(t,x_*)^\lambda\ dx_*\right)\ \Phi(t,x)\ dx \\
&- \int_0^\infty x^{1/\lambda}\ \left(\delta + \Phi(t,x)^\lambda\right) \Phi(t,x)\ dx \\
&+ \int_0^\infty \int_0^x x_*^{(1-\lambda)/\lambda}\ \left(2\delta + \Phi(t,x)^\lambda + \Phi(t,x_*)^\lambda\right)\ \Phi(t,x)\ dx_* dx \\
&= \frac{\mathcal{L}(t)}{\lambda} + \left(2\lambda - 1 - \frac{1}{\lambda}\right)\ \delta\ \int_0^\infty x^{1/\lambda}\ \Phi(t,x)\ dx \\
&+ \int_0^\infty \int_0^x \left(x_*^{(1-\lambda)/\lambda} - \frac{x^{(1-\lambda)/\lambda}}{\lambda}\right)\ \Phi(t,x_*)^\lambda\ \Phi(t,x)\ dx_* dx \\
&- (1-\lambda)\ \int_0^\infty x^{1/\lambda}\ \Phi(t,x)^{1+\lambda}\ dx\,.
\end{aligned}$$

Since $\lambda \in (0,1)$, we end up with

$$\frac{d\mathcal{L}}{dt}(t) \le \frac{\mathcal{L}(t)}{\lambda} - \frac{1-\lambda}{\lambda}\ \int_0^\infty x^{(1-\lambda)/\lambda}\ \int_0^x \Phi(t,x_*)^\lambda\ dx_*\ \Phi(t,x)\ dx\,.$$

Using Lemma A.1, we further obtain

$$\begin{aligned}
\frac{d\mathcal{L}}{dt}(t) &\le \frac{\mathcal{L}(t)}{\lambda} - \frac{1-\lambda}{\lambda^{1+\lambda}}\ \int_0^\infty x^{(1-\lambda)/\lambda}\ \Phi(t,x)\ \left(\int_0^x x_*^{(1-\lambda)/\lambda}\ \Phi(t,x_*)\ dx_*\right)^\lambda\ dx \\
&\le \frac{\mathcal{L}(t)}{\lambda} - \frac{1-\lambda}{(1+\lambda)\lambda^{1+\lambda}}\ \mathcal{L}(t)^{1+\lambda}\,,
\end{aligned}$$

whence (3.6) by the comparison principle. $\square$

We are now in a position to construct stationary solutions to (1.16).

Proposition 3.4 *Given $\varrho > 0$ and $\delta \in (0,1)$, there exists a non-negative and non-increasing function $\Psi_\delta \in L^1(0,\infty) \cap L^\infty(0,\infty)$ such that Supp $\Psi_\delta \subset [0,\gamma/\delta]$,*

$$\|\Psi_\delta\|_1 = \varrho\,, \quad \|\Psi_\delta\|_\infty \le A(\lambda,\varrho) + 2\ \varrho\ \delta\,, \quad \int_0^\infty x^{(1-\lambda)/\lambda}\ \Psi_\delta(x)\ dx \le B(\lambda)\,, \tag{3.7}$$

and

$$\begin{aligned}
& x\ \left(\gamma - \delta\ x - \int_0^x \Psi_\delta(x_*)^\lambda\ dx_*\right)\ \Psi_\delta(x) \\
&= \int_x^\infty \int_0^x \left(\Psi_\delta(x_*)^\lambda + \Psi_\delta(x')^\lambda + 2\delta\right)\ \Psi_\delta(x_*)\ dx' dx_*
\end{aligned} \tag{3.8}$$

for almost every $x \in (0,\gamma/\delta)$. The parameters $A(\lambda,\varrho)$ and $B(\lambda)$ are given by

$$A(\lambda,\varrho) := \left(\frac{2\varrho}{\gamma}\right)^{\gamma}, \quad B(\lambda) := \lambda \ \left(\frac{1+\lambda}{1-\lambda}\right)^{1/\lambda}.$$

Proof. Given $\varrho > 0$ and $\delta \in (0,1)$, we introduce the set $\mathcal{K}_{\varrho,\delta}$ defined by

$$\mathcal{K}_{\varrho,\delta} := \left\{ \begin{array}{l} U \in L^1(0,\infty) \text{ is a non-negative and non-increasing com-} \\ \text{pactly supported function such that } \operatorname{Supp} U \subset [0,\gamma/\delta], \\ \|U\|_1 = \varrho,\ \|U\|_\infty \le z(\delta) \text{ and } \displaystyle\int_0^\infty x^{(1-\lambda)/\lambda}\, U(x)\, dx \le B(\lambda) \end{array} \right\}$$

where $z(\delta)$ is the unique positive zero of $z \longmapsto 2(\delta + z^\lambda)\varrho - \gamma z$. We note that

$$z(0) = A(\lambda,\varrho) \le z(\delta) \le A(\lambda,\varrho) + 2\varrho\delta\,. \tag{3.9}$$

Then, $\mathcal{K}_{\varrho,\delta}$ is a closed convex subset of $L^1(0,\infty)$. In addition, if $(U_n)_{n\ge1}$ is a sequence in $\mathcal{K}_{\varrho,\delta}$, then $(U_n)_{n\ge1}$ is bounded in $BV(0,\infty)$ and there are a subsequence of $(U_n)_{n\ge1}$ (not relabeled) and a function U such that $(U_n(x))$ converges towards $U(x)$ for almost every $x \in (0,\infty)$ as $n \to \infty$. On the one hand, this convergence and the Fatou lemma imply that U is a non-negative and non-increasing function with compact support in $[0,\gamma/\delta]$ and satisfies

$$\|U\|_\infty \le z(\delta) \quad \text{and} \quad \int_0^\infty x^{(1-\lambda)/\lambda}\, U(x)\, dx \le B(\lambda)\,.$$

On the other hand, since $(U_n)_{n\ge1}$ is bounded in $L^\infty(0,\infty)$ with $\operatorname{Supp} U_n \subset [0,\gamma/\delta]$, we deduce from the Lebesgue dominated convergence theorem that (U_n) converges towards U in $L^1(0,\infty)$, whence $\|U\|_1 = \varrho$. Therefore, $U \in \mathcal{K}_{\varrho,\delta}$ and we have thus shown that $\mathcal{K}_{\varrho,\delta}$ is a closed convex and compact subset of $L^1(0,\infty)$.

We now claim that, if $\Phi_0 \in \mathcal{K}_{\varrho,\delta}$, then $\Phi(t,.) \in \mathcal{K}_{\varrho,\delta}$ for each $t \ge 0$, Φ being the corresponding solution to (1.16), (1.17). Indeed, consider $\Phi_0 \in \mathcal{K}_{\varrho,\delta}$. From the analysis of the previous section and (3.1), we know that $\Phi(t,.)$ is a non-negative and non-increasing function in $L^1(0,\infty)$ with $\|\Phi(t)\|_1 = \|\Phi_0\|_1 = \varrho$ for $t \ge 0$. It next readily follows from (3.3) that $m(t) \le \|\Phi_0\|_\infty \vee z(\delta)$, whence $\|\Phi(t)\|_\infty \le z(\delta)$ for $t \ge 0$ by Lemma 3.1. Similarly, as $\operatorname{Supp} \Phi_0 \subset [0,\gamma/\delta]$, the function R defined by (3.5) is bounded from above by $R(0) \vee (\gamma/\delta) = \gamma/\delta$ and we infer from Lemma 3.2 that $\operatorname{Supp} \Phi(t,.) \subset [0,\gamma/\delta]$ for $t \ge 0$. Finally, Lemma 3.3 implies that

$$\int_0^\infty x^{(1-\lambda)/\lambda}\, \Phi(t,x)\, dx \le \left(\int_0^\infty x^{(1-\lambda)/\lambda}\, \Phi_0(x)\, dx\right) \vee B(\lambda) = B(\lambda)\,, \quad t \ge 0\,.$$

Consequently, $\mathcal{K}_{\varrho,\delta}$ is a closed convex and compact subset of $L^1(0,\infty)$ which is left invariant by the semi-flow associated to (1.16), (1.17). Applying Theorem 1.2 with $X = L^1(0,\infty)$ and $K = \mathcal{K}_{\varrho,\delta}$, we obtain the existence of a stationary solution Ψ_δ to (1.16) which belongs to $\mathcal{K}_{\varrho,\delta}$. This last property and (3.9) yield the bounds (3.7) while (3.8) follows from (2.2). $\square$

We are thus left to pass to the limit as $\delta \to 0$ to construct a stationary solution to (1.15) and this is the purpose of the next proposition.

Proposition 3.5 *Given $\varrho > 0$, there exists a non-negative and non-increasing function $\Psi \in L^1(0,\infty) \cap L^\infty(0,\infty)$ such that*

$$\|\Psi\|_1 = \varrho\,, \quad \|\Psi\|_\infty \le A(\lambda,\varrho)\,, \quad \int_0^\infty x^{(1-\lambda)/\lambda}\ \Psi(x)\ dx \le B(\lambda)\,, \tag{3.10}$$

and

$$x\ \left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_*\right)\ \Psi(x) = \int_x^\infty \int_0^x \left(\Psi(x_*)^\lambda + \Psi(x')^\lambda\right)\ \Psi(x_*)\ dx' dx_* \tag{3.11}$$

for almost every $x \in (0,\infty)$, the parameters $A(\lambda,\varrho)$ and $B(\lambda)$ being defined in Proposition 3.4.

Proof. For $\delta \in (0,1)$, let Ψ_δ be a stationary solution to (1.16) given by Proposition 3.4. We infer from Proposition 3.4 that $(\Psi_\delta)_{\delta\in(0,1)}$ is bounded in $BV(0,\infty) \cap L^1(0,\infty; x^{(1-\lambda)/\lambda}\ dx) \cap L^\infty(0,\infty)$ which is clearly compactly embedded in $L^1(0,\infty)$ as $(1-\lambda)/\lambda > 0$. Consequently, there is a sequence $(\delta_n)_{n\ge1}$, $\delta_n \to 0$, and $\Psi \in L^1(0,\infty)$ such that Ψ_{δ_n} converges to Ψ in $L^1(0,\infty)$ and almost everywhere in $(0,\infty)$. Passing to the limit as $\delta_n \to 0$ in (3.7), we obtain (3.10) (with the help of the Fatou lemma for the moment estimate and a weak convergence argument for the L^∞-bound). Owing to the boundedness of (Ψ_{δ_n}) and the convergence of (Ψ_{δ_n}) towards Ψ in $L^1(0,\infty)$ and a.e., it is straightforward to pass to the limit as $\delta_n \to 0$ in (3.8) and obtain (3.11). $\square$

4 Properties of stationary solutions to (1.15)

We next turn to the study of properties of stationary solutions Ψ to (1.15) given by Proposition 3.5. We first establish their $\mathcal{C}^1$-smoothness on $[0,\infty)$.

Proposition 4.1 *Consider $\varrho > 0$ and let Ψ be a stationary solution to (1.15) given by Proposition 3.5. Then, $\Psi \in \mathcal{C}^1([0,\infty))$ and $\Psi^\lambda \in L^1(0,\infty)$ with*

$$L_\lambda := \|\Psi^\lambda\|_1 < \gamma\,. \tag{4.1}$$

Proof. We first observe that, since $\Psi \in L^\infty(0,\infty)$, we have

$$x_0 := \sup\left\{x \ge 0 \ \text{ such that } \ \int_0^x \Psi(x_*)^\lambda\ dx_* < \gamma\right\} \in (0,\infty]\,.$$

It then clearly follows from (3.10) and (3.11) that $\Psi \in \mathcal{C}^1((0,x_0))$ with

$$\begin{aligned} & x\ \left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_*\right)\ \frac{d\Psi}{dx}(x) \\ = & \ -\gamma\ \Psi(x) + \int_x^\infty \Psi(x_*)^{\lambda+1}\ dx_* + \Psi(x)^\lambda\ \int_x^\infty \Psi(x_*)\ dx_* \end{aligned} \tag{4.2}$$

for $x \in (0,x_0)$.

Assume for contradiction that $x_0 < \infty$. Then

$$\int_0^{x_0} \Psi(x_*)^\lambda\ dx_* = \gamma \tag{4.3}$$

and the non-negativity of Ψ and (3.11) imply that $\Psi(x)=0$ for $x > x_0$. It next follows from (4.2) and the monotonicity of Ψ that, if $x\in(0,x_0)$, we have

$$\begin{aligned} x\ \left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_* \right)\ \frac{d\Psi}{dx}(x) &\le -\gamma\ \Psi(x) + 2\ \Psi(x)\ \int_x^{x_0} \Psi(x_*)^\lambda\ dx_* \\ &\le -\gamma\ \Psi(x) + 2\ \Psi(x)\ \left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_* \right)\,, \end{aligned}$$

whence

$$x\ \frac{d\Psi}{dx}(x) \le -\gamma\ \Psi(x)\ \left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_* \right)^{-1} + 2\ \Psi(x)\,.$$

Therefore

$$\begin{aligned} \frac{d}{dx}\left\{ x\ \Psi(x)^\lambda \right\} &= \Psi(x)^\lambda + \lambda\ x\ \Psi(x)^{\lambda-1}\ \frac{d\Psi}{dx}(x) \\ &\le \Psi(x)^\lambda - \gamma\ \lambda\ \Psi(x)^\lambda\ \left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_* \right)^{-1} + 2\ \lambda\ \Psi(x)^\lambda \\ &\le \frac{d}{dx}\left\{ -(1+2\lambda)\ \int_x^{x_0} \Psi(x_*)^\lambda\ dx_* + \lambda\ \gamma\ \log\left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_* \right) \right\}\,, \end{aligned}$$

from which we deduce after integration over $(0,x)$, $x\in(0,x_0)$

$$\begin{aligned} & x\ \Psi(x)^\lambda + (1+2\lambda)\ \int_x^{x_0} \Psi(x_*)^\lambda\ dx_* - \lambda\ \gamma\ \log\left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_* \right) \\ \le\ & (1+2\lambda)\ \int_0^{x_0} \Psi(x_*)^\lambda\ dx_* - \lambda\ \gamma\ \log\gamma \\ \le\ & (1+2\lambda)\ \gamma - \lambda\ \gamma\ \log\gamma\,. \end{aligned} \tag{4.4}$$

But the right-hand side of (4.4) is finite while the left-hand side of (4.4) diverges to infinity as $x \to x_0$ by (4.3), and a contradiction. Therefore, $x_0 = \infty$, $\Psi \in \mathcal{C}^1((0,\infty))$ and

$$\int_0^x \Psi(x_*)^\lambda\ dx_* < \gamma \quad \text{for each} \quad x \in (0,\infty)\,.$$

In particular, $\Psi^\lambda \in L^1(0,\infty)$ and $L_\lambda = \|\Psi^\lambda\|_1 \le \gamma$.

Suppose now for contradiction that $\|\Psi^\lambda\|_1 = \gamma$. Arguing as before, we realize that (4.4) is valid for every $x\in(0,\infty)$. Then, letting $x\to\infty$ in (4.4) yields that the left-hand side of (4.4) diverges to infinity while the right-hand side is finite, whence a contradiction. We have thus shown that $L_\lambda < \gamma$, whence the claim (4.1).

We next turn to the regularity of Ψ at $x=0$. Since Ψ is a non-increasing and bounded function, the limit $\Psi(0+)$ of $\Psi(x)$ as $x \to 0$ exists and is finite and we may actually set $\Psi(0)=\Psi(0+)$. Then $\Psi \in \mathcal{C}([0,\infty))$. In addition, since $\|\Psi\|_1 = \varrho > 0$, Ψ is not identically equal to zero and thus $\Psi(0)>0$ by the monotonicity of Ψ. Since the identity (3.11) also reads

$$\left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_*\right)\ \Psi(x) = \int_x^\infty \Psi(x_*)^{\lambda+1}\ dx_* + \frac{1}{x}\ \int_0^x \Psi(x_*)^\lambda\ dx_*\ \int_x^\infty \Psi(x_*)\ dx_*\,,$$

we may let $x \to 0$ in the previous inequality and obtain

$$\gamma\ \Psi(0) = \int_0^\infty \Psi(x_*)^{\lambda+1}\ dx_* + \varrho\ \Psi(0)^\lambda\,. \tag{4.5}$$

It follows from (3.11), (4.5) and the monotonicity of Ψ that

$$\begin{aligned}
\gamma\ (\Psi(0)-\Psi(x)) &= \int_0^x \Psi(x_*)^{1+\lambda}\ dx_* - \Psi(x)\ \int_0^x \Psi(x_*)^\lambda\ dx_* \\
&+ \varrho\ \Psi(0)^\lambda - \frac{1}{x}\ \int_0^x \Psi(x_*)^\lambda\ dx_*\ \int_x^\infty \Psi(x_*)\ dx_* \\
&\le 2\ x\ \Psi(0)^{1+\lambda} + \varrho\ \left(\Psi(0)^\lambda - \frac{1}{x}\ \int_0^x \Psi(x_*)^\lambda\ dx_*\right) \\
&\le 2\ x\ \Psi(0)^{1+\lambda} + \varrho\ \left(\Psi(0)^\lambda - \Psi(x)^\lambda\right) \\
&\le 2\ x\ \Psi(0)^{1+\lambda} + \varrho\ \lambda\ \Psi(x)^{\lambda-1}\ (\Psi(0)-\Psi(x))\,,
\end{aligned}$$

whence

$$\left(\gamma - \varrho\ \lambda\ \Psi(x)^{\lambda-1}\right)\ \frac{\Psi(0)-\Psi(x)}{x} \le 2\ \Psi(0)^{1+\lambda}\,.$$

As $\lambda \in (0,1)$ and $\gamma \ge \varrho\ \Psi(0)^{\lambda-1}$ by (4.5), we have $\gamma > \lambda\ \varrho\ \Psi(0)^{\lambda-1}$ and we infer from the continuity of Ψ at $x=0$ that there are $\delta_1>0$ and $x_1>0$ such that

$$\gamma - \varrho\ \lambda\ \Psi(x)^{\lambda-1} \ge \delta_1 \quad \text{for} \quad x \in (0,x_1)\,.$$

Combining the above two inequalities and the monotonicity of Ψ yield

$$0 \le \frac{\Psi(0)-\Psi(x)}{x} \le \frac{2}{\delta_1}\ \Psi(0)^{1+\lambda} \quad \text{for} \quad x \in (0,x_1)\,, \tag{4.6}$$

so that $x \longmapsto (\Psi(0)-\Psi(x))/x$ belongs to $L^\infty(0,x_1)$.

Another consequence of (4.5) is that we may pass to the limit as $x \to 0$ in (4.2) to deduce that

$$\lim_{x\to 0} x\ \frac{d\Psi}{dx}(x) = 0\,. \tag{4.7}$$

Using once more (4.5), the identity (4.2) also reads

$$\begin{aligned}\gamma\ x\ \frac{d\Psi}{dx}(x) \quad &= \quad x\ \int_0^x \Psi(x_*)^\lambda\ dx_*\ \frac{d\Psi}{dx}(x) - \gamma\ (\Psi(x) - \Psi(0)) - \int_0^x \Psi(x_*)^{1+\lambda}\ dx_* \\ &+ \quad \varrho\ \left(\Psi(x)^\lambda - \Psi(0)^\lambda\right) - \Psi(x)^\lambda\ \int_0^x \Psi(x_*)\ dx_*\,,\end{aligned}$$

from which we deduce that

$$\begin{aligned}&\gamma\ \frac{d\Psi}{dx}(x) + \left(\gamma - \lambda\ \varrho\ \Psi(0)^{\lambda-1}\right)\ \frac{\Psi(x) - \Psi(0)}{x} \\ = \quad &\left(\frac{1}{x}\ \int_0^x \Psi(x_*)^\lambda\ dx_*\right)\ \left(x\ \frac{d\Psi}{dx}(x)\right) - \frac{1}{x}\ \int_0^x \Psi(x_*)^{1+\lambda}\ dx_* - \frac{\Psi(x)^\lambda}{x}\ \int_0^x \Psi(x_*)\ dx_* \\ + \quad &\frac{\varrho}{x}\ \left(\Psi(x)^\lambda - \Psi(0)^\lambda - \lambda\ \Psi(0)^{\lambda-1}(\Psi(x) - \Psi(0))\right)\ ,\end{aligned}$$

On the one hand, by (4.7) and the continuity of Ψ at $x = 0$, we have

$$\begin{aligned}&\lim_{x\to 0} \left(\frac{1}{x}\ \int_0^x \Psi(x_*)^\lambda\ dx_*\right)\ \left(x\ \frac{d\Psi}{dx}(x)\right) = 0\,, \\ &\lim_{x\to 0} \frac{1}{x}\ \int_0^x \Psi(x_*)^{1+\lambda}\ dx_* = \lim_{x\to 0} \frac{\Psi(x)^\lambda}{x}\ \int_0^x \Psi(x_*)\ dx_* = \Psi(0)^{1+\lambda}\,.\end{aligned}$$

On the other hand, it follows from the concavity of $r \mapsto r^\lambda$, the monotonicity of Ψ and (4.6) that, if $x \in (0, x_1)$,

$$\begin{aligned}&\left|\frac{1}{x}\ \left(\Psi(x)^\lambda - \Psi(0)^\lambda - \lambda\ \Psi(0)^{\lambda-1}(\Psi(x) - \Psi(0))\right)\right| \\ = \quad &\frac{1}{x}\ \left(\Psi(0)^\lambda - \Psi(x)^\lambda - \lambda\ \Psi(0)^{\lambda-1}(\Psi(0) - \Psi(x))\right) \\ \le \quad &\frac{\lambda}{x}\ \left(\Psi(x)^{\lambda-1} - \Psi(0)^{\lambda-1}\right)\ (\Psi(0) - \Psi(x)) \\ \le \quad &\lambda\ \sup_{x_*\in(0,x_1)} \left(\frac{\Psi(0) - \Psi(x_*)}{x_*}\right)\ \left(\Psi(x)^{\lambda-1} - \Psi(0)^{\lambda-1}\right) \\ \le \quad &\frac{2\ \lambda}{\delta_1}\ \Psi(0)^{1+\lambda}\ \left(\Psi(x)^{\lambda-1} - \Psi(0)^{\lambda-1}\right) \xrightarrow[x\to 0]{} 0\end{aligned}$$

by the continuity of Ψ at $x = 0$. Consequently,

$$\lim_{x\to 0} \left\{\gamma\ \frac{d\Psi}{dx}(x) + \left(\gamma - \lambda\ \varrho\ \Psi(0)^{\lambda-1}\right)\ \frac{\Psi(x) - \Psi(0)}{x}\right\} = -2\ \Psi(0)^{1+\lambda}\,. \tag{4.8}$$

Introducing $\omega := 1 - \lambda\ \varrho\ \Psi(0)^{\lambda-1}\ \gamma^{-1}$, we have $\omega > 0$ by (4.5) and the previous limit also reads

$$\lim_{x\to 0} \left\{x^{-\omega}\ \frac{d}{dx}\left(x^\omega\ (\Psi(x) - \Psi(0))\right)\right\} = -\frac{2\ \Psi(0)^{1+\lambda}}{\gamma}\,,$$

whence, by integration,

$$\lim_{x\to 0} \frac{\Psi(x)-\Psi(0)}{x} = -\frac{2\ \Psi(0)^{1+\lambda}}{\gamma\ (\omega+1)}\,.$$

Therefore, Ψ is differentiable at $x=0$ with

$$\frac{d\Psi}{dx}(0) = -\frac{2\ \Psi(0)^{1+\lambda}}{2\ \gamma - \lambda\ \varrho\ \Psi(0)^{\lambda-1}} < 0\,, \tag{4.9}$$

and (4.8) ensures the continuity of $d\Psi/dx$ at $x=0$. $\square$

We next turn to the positivity and monotonicity properties of stationary solutions to (1.15).

Proposition 4.2 *Consider $\varrho > 0$ and let Ψ be a stationary solution to (1.15) given by Proposition 3.5. Then*

$$\Psi(x) > 0 \ \ \textit{and} \ \ \frac{d\Psi}{dx}(x) < 0 \ \ \textit{for} \ \ \ x \ge 0\,. \tag{4.10}$$

Proof. Recalling that $L_\lambda := \|\Psi^\lambda\|_1 < \gamma$ by Proposition 4.1, we have for $\delta > 0$ and $x \ge \delta$

$$\delta\ (\gamma - L_\lambda) \le x\ \left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_* \right)\,.$$

We then infer from (4.2) and the non-positivity of $d\Psi/dx \le 0$ that

$$\delta\ (\gamma - L_\lambda)\ \frac{d\Psi}{dx}(x) \ge -\gamma\ \Psi(x)\,, \quad x \ge \delta\,.$$

Therefore,

$$\Psi(x) \ge \Psi(\delta)\ \exp\left\{ \frac{\gamma\ (\delta - x)}{\delta\ (\gamma - L_\lambda)} \right\}\,, \quad x \ge \delta\,.$$

Owing to the continuity of Ψ at $x=0$ and the positivity of $\Psi(0)$, we also have $\Psi(\delta) > 0$ for δ sufficiently small, which, together with the above lower bound for Ψ, entail the positivity of Ψ in $[0,\infty)$. Similarly, it follows from (4.2) that Ψ is twice differentiable in $(0,\infty)$ with

$$\begin{aligned}
& x\ \left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_* \right)\ \frac{d^2\Psi}{dx^2}(x) \\
= & \left(x\ \Psi(x)^\lambda + \lambda\ \Psi(x)^{\lambda-1}\ \int_x^\infty \Psi(x_*)\ dx_* + \int_0^x \Psi(x_*)^\lambda\ dx_* - 2\ \gamma \right)\ \frac{d\Psi}{dx}(x) \\
- & \ 2\ \Psi(x)^{1+\lambda} \\
\le & \ -2\ \gamma\ \frac{d\Psi}{dx}(x)\,,
\end{aligned}$$

the last inequality being a consequence of the positivity and monotonicity of Ψ. Consequently, if $\delta > 0$ and $x \geq \delta$,

$$\begin{aligned}
\frac{d^2\Psi}{dx^2}(x) &\leq -\frac{2\,\gamma}{x}\,\frac{d\Psi}{dx}(x)\,\left(\gamma - \int_0^x \Psi(x_*)^\lambda\,dx_*\right)^{-1} \\
&\leq -\frac{2\,\gamma}{\delta\,(\gamma - L_\lambda)}\,\frac{d\Psi}{dx}(x)\,,
\end{aligned}$$

whence

$$\frac{d\Psi}{dx}(x) \leq \frac{d\Psi}{dx}(\delta)\,\exp\left\{\frac{2\,\gamma\,(\delta - x)}{\delta\,(\gamma - L_\lambda)}\right\}, \qquad x \geq \delta\,.$$

Recalling that $d\Psi/dx \in \mathcal{C}([0,\infty))$ with $d\Psi(0)/dx < 0$ by (4.9), we easily deduce from the previous inequality that $d\Psi(x)/dx < 0$ for $x \geq 0$. $\square$

We finally identify the behaviour of Ψ as $x \to \infty$.

Proposition 4.3 *Consider $\varrho > 0$ and let Ψ be a stationary solution to (1.15) given by Proposition 3.5. Then $L_\lambda := \left\|\Psi^\lambda\right\|_1 > 1$ and there is a positive constant $b > 0$ such that*

$$\lim_{x\to\infty} x^\alpha\,\Psi(x) = b \quad \textit{and} \quad \lim_{x\to\infty} x^{1+\alpha}\,\frac{d\Psi}{dx}(x) = -\alpha\,b\,, \tag{4.11}$$

with $\alpha := \gamma/(\gamma - L_\lambda) > 0$.

Proof. We first establish that

$$L_\lambda = \left\|\Psi^\lambda\right\|_1 > 1\,. \tag{4.12}$$

Indeed, since $\Psi(x) > 0$ for $x \geq 0$, we multiply (4.2) by $\lambda\,\Psi(x)^{\lambda-1}$ and integrate over $(0,\infty)$ to obtain

$$\begin{aligned}
&\left[x\,\left(\gamma - \int_0^x \Psi(x_*)^\lambda\,dx_*\right)\,\Psi(x)^\lambda\right]_0^\infty - \int_0^\infty \Psi(x)^\lambda\,\left(\gamma - \int_0^x \Psi(x_*)^\lambda\,dx_* - x\,\Psi(x)^\lambda\right)\,dx \\
&= -\lambda\,\gamma\,L_\lambda + \lambda\,\int_0^\infty \int_x^\infty \left(\Psi(x)^{\lambda-1}\,\Psi(x_*)^{\lambda+1} + \Psi(x)^{2\lambda-1}\,\Psi(x_*)\right)\,dx_*dx \\
&= -\lambda\,\gamma\,L_\lambda + \lambda\,\int_0^\infty \int_0^x \left(\Psi(x)^{\lambda+1}\,\Psi(x_*)^{\lambda-1} + \Psi(x)\,\Psi(x_*)^{2\lambda-1}\right)\,dx_*dx\,.
\end{aligned}$$

Since Ψ is non-increasing and $\Psi^\lambda \in L^1(0,\infty)$, the boundary terms vanish and we end up with

$$\begin{aligned}
(1-\lambda)\,\gamma\,L_\lambda &= \int_0^\infty \int_0^x \left(\Psi(x)^\lambda\,\Psi(x_*)^\lambda + \Psi(x)^{2\lambda}\right)\,dx_*dx \\
&- \lambda\,\int_0^\infty \int_0^x \left(\Psi(x)^{\lambda+1}\,\Psi(x_*)^{\lambda-1} + \Psi(x)\,\Psi(x_*)^{2\lambda-1}\right)\,dx_*dx
\end{aligned}$$

$$L_\lambda = \int_0^\infty \int_0^x \left(\Psi(x)^\lambda + \Psi(x_*)^\lambda\right) \left(\Psi(x)^{\lambda-1} - \lambda\ \Psi(x_*)^{\lambda-1}\right)\ \Psi(x)\ dx_* dx$$

Since $\Psi(x) \le \Psi(x_*)$ for $x_* \in (0,x)$, it follows from Lemma A.2 and the elementary inequality $\Psi(x)^\lambda + \Psi(x_*)^\lambda \le 2^{1-\lambda}\ (\Psi(x) + \Psi(x_*))^\lambda$ that

$$L_\lambda \le 2^{1-\lambda} \int_0^\infty \int_0^x \Psi(x)^\lambda\ \Psi(x_*)^\lambda dx_* dx = 2^{-\lambda}\ L_\lambda^2\,.$$

Therefore, $L_\lambda \ge 2^\lambda > 1$ which completes the proof of (4.12).

We next infer from (4.2) and the positivity and monotonicity of Ψ that

$$x\ (\gamma - L_\lambda)\ \frac{d\Psi}{dx}(x) \ge x\ \left(\gamma - \int_0^x \Psi(x_*)^\lambda\ dx_*\right)\ \frac{d\Psi}{dx}(x) \ge -\gamma\ \Psi(x)\,,$$

whence

$$x\ \left|\frac{d\Psi}{dx}(x)\right| \le \frac{\gamma}{\gamma - L_\lambda}\ \Psi(x)\,,$$

and

$$(\gamma - L_\lambda)\ x\ \frac{d\Psi}{dx}(x) + \gamma\ \Psi(x) \le \left(2\ \Psi(x) + x\ \left|\frac{d\Psi}{dx}(x)\right|\right)\ \int_x^\infty \Psi(x_*)^\lambda\ dx_*$$

for $x > 0$. We combine the previous inequalities to obtain

$$(\gamma - L_\lambda)\ x\ \frac{d\Psi}{dx}(x) + \gamma\ \Psi(x) \le \left(2 + \frac{\gamma}{\gamma - L_\lambda}\right)\ \Psi(x)\ \int_x^\infty \Psi(x_*)^\lambda\ dx_*\,.$$

Recall that $\alpha = \gamma/(\gamma - L_\lambda) > 0$ and fix $\varepsilon \in (0,\alpha)$. Since $\Psi^\lambda \in L^1(0,\infty)$, there is $x_\varepsilon > 0$ such that

$$\int_x^\infty \Psi(x_*)^\lambda\ dx_* \le \frac{\gamma - L_\lambda}{2+\alpha}\ \varepsilon \quad \text{for} \quad x \ge x_\varepsilon\,.$$

Therefore, for $x \ge x_\varepsilon$, we have

$$x\ \frac{d\Psi}{dx}(x) + \alpha\ \Psi(x) \le \varepsilon\ \Psi(x)\,,$$

from which we deduce by integration that

$$\Psi(x) \le \frac{x_\varepsilon^{\alpha-\varepsilon}\ \Psi(x_\varepsilon)}{x^{\alpha-\varepsilon}} \quad \text{for} \quad x \ge x_\varepsilon\,.$$

Recalling that $\Psi \in L^\infty(0,\infty)$, we have thus established that, for each $\varepsilon \in (0,\alpha)$, there is $\kappa_\varepsilon > 0$ such that

$$\Psi(x) \le \kappa_\varepsilon\ x^{-\alpha+\varepsilon} \quad \text{for} \quad x > 0\,. \tag{4.13}$$

We use once more (4.2) to obtain that

$$
\begin{aligned}
(\gamma - L_\lambda)\ \frac{d}{dx}\left\{x^\alpha\ \Psi(x)\right\} &= (\gamma - L_\lambda)\ x^\alpha\ \frac{d\Psi(x)}{dx} + \gamma\ x^{\alpha-1}\ \Psi(x) \\
&= x^{\alpha-1}\ \left(\int_x^\infty \Psi(x_*)^{1+\lambda}\ dx_* + \Psi(x)^\lambda\ \int_x^\infty \Psi(x_*)\ dx_*\right) \\
&- x^\alpha\ \frac{d\Psi}{dx}(x)\ \int_x^\infty \Psi(x_*)^\lambda\ dx_*\,. \qquad (4.14)
\end{aligned}
$$

Now, since $L_\lambda > 1$ by (4.12), we have $\alpha > \lambda\alpha > 1$ and there exists $\varepsilon \in (0, \alpha - 1)$ such that $\lambda\alpha > 1 + (1+\lambda)\varepsilon$. Then, on the one hand, by the monotonicity of Ψ and (4.13), an integration by parts yields

$$
\begin{aligned}
0 &\le -\int_1^\infty x^\alpha\ \frac{d\Psi}{dx}(x)\ \int_x^\infty \Psi(x_*)^\lambda\ dx_* dx \\
&\le \Psi(1)\ \int_1^\infty \Psi(x_*)^\lambda\ dx_* + \int_1^\infty \Psi(x)\ \left(\alpha\ x^{\alpha-1}\ \int_x^\infty \Psi(x_*)^\lambda\ dx_* - x^\alpha\ \Psi(x)^\lambda\ \right)\ dx \\
&\le C + \frac{\alpha\ \kappa_\varepsilon^{1+\lambda}}{\lambda\alpha - 1 - \lambda\varepsilon}\ \int_1^\infty x^{(1+\lambda)\varepsilon - \lambda\alpha}\ dx \\
&\le C(\varepsilon)\,.
\end{aligned}
$$

On the other hand, it follows from the monotonicity of Ψ and (4.13) that

$$
\begin{aligned}
&\int_1^\infty x^{\alpha-1}\ \left(\int_x^\infty \Psi(x_*)^{1+\lambda}\ dx_* + \Psi(x)^\lambda\ \int_x^\infty \Psi(x_*)\ dx_*\right)\ dx \\
\le\ & 2\ \frac{\kappa_\varepsilon^{1+\lambda}}{\alpha - 1 - \varepsilon}\ \int_1^\infty x^{(1+\lambda)\varepsilon - \lambda\alpha}\ dx \le C(\varepsilon)\,.
\end{aligned}
$$

Consequently, the right-hand side of (4.14) belongs to $L^1(1,\infty)$ and is positive, from which we conclude that $x \longmapsto x^\alpha\ \Psi(x)$ has a positive limit b as $x \to \infty$. We have thus proved the first assertion in (4.11).

As for $d\Psi/dx$, we note that the large x-behaviour of Ψ ensures that there is x_∞ large enough such that $x^\alpha\ \Psi(x) \le 2b$ for $x \ge x_\infty$. Consequently, for $x \ge x_\infty$, we have

$$
\begin{aligned}
x^\alpha\ \left(\int_x^\infty \Psi(x_*)^{\lambda+1}\ dx_* + \Psi(x)^\lambda\ \int_x^\infty \Psi(x_*)\ dx_*\right) &\le 2\ x^\alpha\ \Psi(x)^\lambda\ \int_x^\infty \Psi(x_*)\ dx_* \\
&\le 2\ (2b)^{\lambda+1}\ x^{(1-\lambda)\alpha} \int_x^\infty x_*^{-\alpha}\ dx_* \\
&\le \frac{2\ (2b)^{\lambda+1}}{\alpha - 1}\ x^{1-\lambda\alpha}\,.
\end{aligned}
$$

Recalling that $\lambda\alpha > 1$ by (4.12), we conclude that

$$
\lim_{x\to\infty} x^\alpha\ \left(\int_x^\infty \Psi(x_*)^{\lambda+1}\ dx_* + \Psi(x)^\lambda\ \int_x^\infty \Psi(x_*)\ dx_*\right) = 0\,.
$$

We now multiply (4.2) by x^α and let $x \to \infty$ in the resulting identity with the help of the previous limit and the first statement in (4.11) to complete the proof of (4.11). $\square$

5 Proof of Theorem 1.1

Consider $\varrho \in (0,\infty)$. By Proposition 3.5 there exists a non-negative function $\Psi \in L^1(0,\infty)$ satisfying (3.10) and (3.11). In addition, $\Psi \in \mathcal{C}^1([0,\infty))$ is a positive and decreasing function from $[0,\infty)$ onto $(0,y_0]$ with $y_0 := \Psi(0)$ by Propositions 4.1 and 4.2. We then denote its inverse function by $\Xi : (0,y_0] \longrightarrow [0,\infty)$ which is also a decreasing and non-negative function in $\mathcal{C}^1((0,y_0])$ and put $\xi := -d\Xi/dy > 0$. We extend Ξ and ξ to (y_0,∞) by setting $\Xi(y) = \xi(y) = 0$ for $y > y_0$. Then $\xi \in \mathcal{C}((0,\infty) \setminus \{y_0\})$ with $\xi(y_0-) = -(d\Psi/dx(0))^{-1} > 0$ by (4.9) and $\xi(y_0+) = 0$. Also, since $\Xi(y) \longrightarrow \infty$ as $y \to 0$, we infer from Propositions 4.1 and 4.3 that

$$\lim_{y\to 0}\ y\ \Xi(y)^{\gamma/(\gamma-L_\lambda)} = b\,, \quad \lim_{y\to 0}\ -\frac{\Xi(y)^{1+(\gamma/(\gamma-L_\lambda))}}{\xi(y)} = -\frac{\gamma\ b}{\gamma - L_\lambda}$$

with $L_\lambda := \left\|\Psi^\lambda\right\|_1 \in (1,\gamma)$, whence

$$\lim_{y\to 0}\ y^{2-(L_\lambda/\gamma)}\ \xi(y) = \frac{\gamma - L_\lambda}{\gamma}\ b^{1-L_\lambda/\gamma} > 0\,.$$

In addition, Ξ being a $\mathcal{C}^1$-diffeomorphism from $(0,y_0]$ onto $[0,\infty)$, a simple change of variables yields

$$\int_0^\infty y\ \xi(y)\ dy = \|\Psi\|_1 = \varrho \quad\text{and}\quad \int_0^\infty y^\lambda\ \xi(y)\ dy = \left\|\Psi^\lambda\right\|_1 = L_\lambda\,.$$

We have thus shown that ξ enjoys all the properties (1.9), (1.10) and (1.11) listed in Theorem 1.1. To check (1.12), we consider $y \in (0,y_0)$ and take $x = \Xi(y)$ in (4.2) to obtain

$$-\left(\gamma - \int_0^{\Xi(y)} \Psi(x_*)^\lambda\ dx_*\right)\ \frac{\Xi(y)}{\xi(y)} = -\gamma\ y + \int_{\Xi(y)}^\infty \left(\Psi(x_*)^{\lambda+1} + y^\lambda\ \Psi(x_*)\right)\ dx_*\,,$$

whence (1.12) after performing the change of variables $x_* = \Xi(y_*)$ in the integrals. Finally, let $t \in (0,\infty)$ and $\vartheta \in \mathcal{C}_0^\infty((0,\infty))$. Then

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty g_s(t,y)\ \vartheta(y)\ dy &= \frac{d}{dt}\left(\frac{1}{t^{2\gamma}}\ \int_0^\infty \xi\left(\frac{y}{t^\gamma}\right)\ \vartheta(y)\ dy\right)\\
&= \frac{d}{dt}\left(\frac{1}{t^\gamma}\ \int_0^\infty \xi(y)\ \vartheta\left(y\ t^\gamma\right)\ dy\right)\\
&= -\frac{\gamma}{t^{1+\gamma}}\ \int_0^\infty \xi(y)\ \vartheta\left(y\ t^\gamma\right)\ dy + \frac{\gamma}{t}\ \int_0^\infty y\ \xi(y)\ \partial_y\vartheta\left(y\ t^\gamma\right)\ dy\\
&= \frac{\gamma}{t}\int_0^\infty y\ \xi(y)\ \partial_y\vartheta\left(y\ t^\gamma\right)\ dy - \frac{\gamma}{t}\int_0^\infty \xi(y)\ \int_0^y \partial_y\vartheta\left(y_*\ t^\gamma\right)\ dy_* dy\,,
\end{aligned}$$

whence

$$t \, \frac{d}{dt} \int_0^\infty g_s(t,y) \, \vartheta(y) \, dy = \gamma \, \int_0^\infty \left(y \, \xi(y) - \int_y^\infty \xi(y_*) \, dy_* \right) \partial_y \vartheta \left(y \, t^\gamma \right) \, dy_* dy \, .$$

Using (1.12), we deduce that

$$\begin{aligned} t \, \frac{d}{dt} \int_0^\infty g_s(t,y) \, \vartheta(y) \, dy &= \int_0^\infty \partial_y \vartheta \left(y \, t^\gamma \right) \int_0^y \left(y^\lambda + y_*^\lambda \right) \, y_* \, \xi(y_*) \, dy_* \, \xi(y) \, dy \\ &- \int_0^\infty \partial_y \vartheta \left(y \, t^\gamma \right) \left(\int_y^\infty y_*^\lambda \, \xi(y_*) \, dy_* \right) \left(\int_y^\infty \xi(y') \, dy' \right) \, dy \\ &= \frac{1}{t^{(3+\lambda)\gamma}} \int_0^\infty \partial_y \vartheta(y) \int_0^y \left(y^\lambda + y_*^\lambda \right) \, y_* \, \xi \left(\frac{y_*}{t^\gamma} \right) \, dy_* \, \xi \left(\frac{y}{t^\gamma} \right) \, dy \\ &- \frac{1}{t^\gamma} \int_0^\infty \vartheta \left(y \, t^\gamma \right) \int_y^\infty \left(y^\lambda + y_*^\lambda \right) \, \xi(y_*) \, \xi(y) \, dy_* dy \\ &= t^{(1-\lambda)\gamma} \int_0^\infty \partial_y \vartheta(y) \int_0^y \left(y^\lambda + y_*^\lambda \right) \, y_* \, g_s(t,y_*) \, g_s(t,y) \, dy_* dy \\ &- t^{(1-\lambda)\gamma} \int_0^\infty \vartheta(y) \int_y^\infty \left(y^\lambda + y_*^\lambda \right) \, g_s(t,y_*) \, g_s(t,y) \, dy_* dy \\ &= t \int_0^\infty \int_0^y \left(y_* \, \partial_y \vartheta(y) - \vartheta(y_*) \right) \left(y^\lambda + y_*^\lambda \right) \, g_s(t,y_*) \, g_s(t,y) \, dy_* dy \, , \end{aligned}$$

since $\gamma = 1/(1-\lambda)$. Dividing the above equality by t yields that g_s is a weak solution to (1.2) and completes the proof of Theorem 1.1. □

A Two inequalities

Lemma A.1 *Consider $\vartheta \in (0,1)$ and a non-negative and non-increasing measurable function U such that $U^\vartheta \in L^\infty(0,\infty)$. Then*

$$\int_0^x x_*^{(1-\vartheta)/\vartheta} \, U(x_*) \, dx_* \le \vartheta \, \left(\int_0^x U(x_*)^\vartheta \, dx_* \right)^{1/\vartheta} \tag{A.1}$$

for $x \in (0,\infty)$. Furthermore, if $U^\vartheta \in L^1(0,\infty)$, then $U \in L^1(0,\infty; x^{(1-\vartheta)/\vartheta} \, dx)$ and

$$\int_0^\infty x^{(1-\vartheta)/\vartheta} \, U(x) \, dx \le \vartheta \, \left(\int_0^\infty U(x)^\vartheta \, dx \right)^{1/\vartheta} \, . \tag{A.2}$$

Proof. Consider $x \in (0,\infty)$ and $x_* \in (0,x)$. By the monotonicity of U, we have

$$x_* \, U(x_*)^\vartheta \le \int_0^{x_*} U(y)^\vartheta \, dy \, ,$$

whence

$$x_*^{1/\vartheta}\ U(x_*) \le \left(\int_0^{x_*} U(y)^\vartheta\ dy \right)^{1/\vartheta} .$$

Consequently,

$$\begin{aligned} \int_0^x x_*^{(1-\vartheta)/\vartheta}\ U(x_*)\ dx_* &= \int_0^x x_*^{(1-\vartheta)/\vartheta}\ U(x_*)^{1-\vartheta}\ U(x_*)^\vartheta\ dx_* \\ &\le \int_0^x U(x_*)^\vartheta\ \left(\int_0^{x_*} U(y)^\vartheta\ dy \right)^{(1-\vartheta)/\vartheta}\ dx_* \\ &= \vartheta\ \left(\int_0^x U(x_*)^\vartheta\ dx_* \right)^{1/\vartheta} , \end{aligned}$$

whence (A.1). Next, if $U^\vartheta \in L^1(0,\infty)$, we may let $x \to \infty$ in (A.1) to obtain (A.2) and thus complete the proof of Lemma A.1. □

Lemma A.2 *Consider $\lambda \in [0,1]$, $r > 0$ and $r_* \in (0,r)$. Then*

$$(1-\lambda)\ 2^\lambda\ \frac{r^\lambda\ r_*^{\lambda-1}}{(r+r_*)^\lambda} \le r_*^{\lambda-1} - \lambda\ r^{\lambda-1} \le \frac{r^\lambda\ r_*^{\lambda-1}}{(r+r_*)^\lambda} . \tag{A.3}$$

Proof. The inequalities (A.3) being obvious for $\lambda \in \{0,1\}$, we restrict ourselves to $\lambda \in (0,1)$ and put

$$p(z) := (1+z)^\lambda\ \left(1 - \lambda\ z^{1-\lambda}\right) , \quad q(z) := z^\lambda - z - (1-\lambda)$$

for $z \in (0,1)$. Then

$$p'(z) = \frac{\lambda}{z^\lambda\ (1+z)^{1-\lambda}}\ q(z) , \quad q'(z) = \lambda\ z^{\lambda-1} - 1 ,$$

from which we deduce that $q(z) \le q\left(\lambda^\gamma\right)$ for $z \in (0,1)$. Since $q\left(\lambda^\gamma\right) = \left(\lambda^{\lambda\gamma} - 1\right)\ (1-\lambda) < 0$, we conclude that $p'(z) \le 0$ for $z \in (0,1)$. Consequently, $p(1) \le p(z) \le p(0)$ for $z \in (0,1)$, whence

$$(1-\lambda)\ 2^\lambda \le p(z) \le 1 \quad \text{for} \quad z \in (0,1) .$$

We next consider $r > 0$ and $r_* \in (0,r)$ and take $z = r_*/r$ in the previous inequality to obtain (A.3). □

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