Quantum Gravity, Random Tensors and Renormalization I

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Regards sur la gravité quantique
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Plan of the Mini-Course

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I Quantum Gravity as Random Geometry

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II Random Tensors

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III Tensor Group Field Theories and their Renormalization

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A Recent Reference: Riv., "The Tensor Track, III," arXiv:1311.1461

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Feynman's quantization: functional integral

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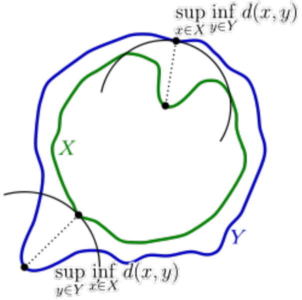

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Remark: Feynman graphs are metric spaces (for the graph distance) hence sum over Feynman graphs can in principle lead to random metric spaces as desired for quantizing gravity.

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Large N behavior studied with $1/N$ expansion

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Invariant interactions: $(\bar{\phi}\cdot\phi)^p$ factorized, hence not connected. For $p = 2$, interacting ϕ^4-type vector model

$$Z_{int}(\lambda, N) = \frac{1}{(2i\pi)^N} \int e^{-(\bar{\phi}\cdot\phi) - \lambda(\bar{\phi}\cdot\phi)^2} d\bar{\phi} d\phi$$

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Propagator: from models to QFT with renormalization group flow

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(Single) Matrix Models have exactly one connected invariant interaction at every (even) degree, namely $Tr(MM^\dagger)^p$.

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Many connected invariant interaction at every (even) degree.

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Z_1^c(n) &= 1, 0, 0, 0, 0, \ldots \\
Z_2^c(n) &= 1, 1, 1, 1, 1, 1, 1, \ldots \\
Z_3^c(n) &= 1, 3, 7, 26, 97, 624, 4163, \ldots \\
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Remark: symmetry of interactions could be interpreted as pre-geometric locality.

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A rank D tensor $\Phi_{i_1,\cdots,i_D}$ can also be viewed as Fourier truncation (N ultraviolet cutoff) of a function of D variables $\Phi(x_1,\cdots,x_D)$, $x_i \in G$ compact Lie group (simplest case: $G = U(1)$...). We can use $-\sum_i \Delta_i^G + m^2$ as propagator (instead of m^2), where Δ^G is the Laplacian on G.

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If we apply this idea to matrix models we obtain non-commutative quantum field theory. If we apply this idea to tensor models we obtain a higher category of quantum field theories which we call tensorial group field theories.

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The second space can be seen as a set of random labels living on the first.

The Continuous Random Tree

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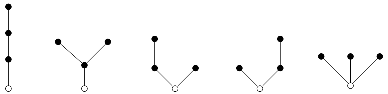
The number of plane trees with n edges is Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$: 1, 2, 5, 14, 42....



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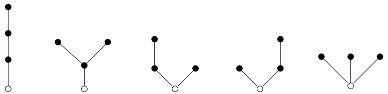

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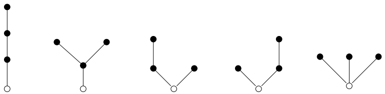

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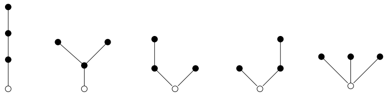

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The equidistributed measure on plane trees converges (in Gromov-Hausdorff sense) to a universal object as $n \to \infty$, namely the Continuous Random Tree (CRT).

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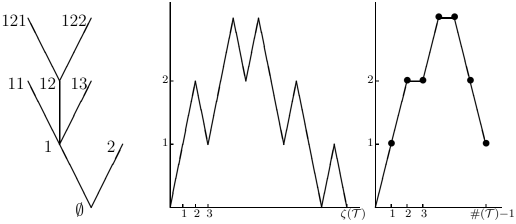

tree **t** contour function height function

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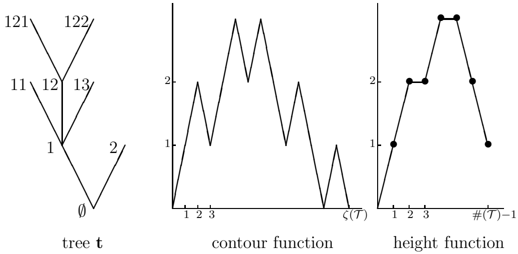

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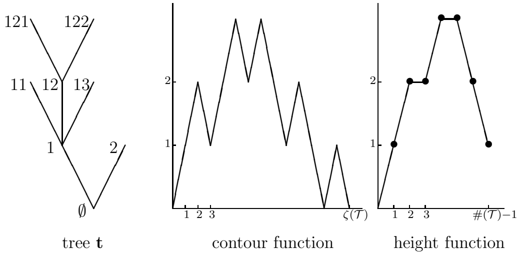

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$$d_{Hausdorff} = 2, \; d_{spectral=4/3}.$$

CRT and Vector Models

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Simplest interacting complex ϕ^4 vector model: conjugate vector fields $(\phi = \{\phi_i\}, \bar\phi = \{\bar\phi_i\}, i = 1, \cdots, N$, $\lambda(\bar\phi \cdot \phi)^2$ interaction.

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$$\begin{aligned} Z(\lambda, N) &= \frac{1}{(2i\pi)^N} \int d\sigma \frac{e^{-\sigma^2/2}}{\sqrt{2\pi}} \int e^{-(\bar\phi\cdot\phi) + i\sqrt{2\lambda}(\bar\phi\cdot\phi)\sigma} d\bar\phi d\phi \\ &= \int \frac{d\sigma}{\sqrt{2\pi}} e^{-\sigma^2/2 - N\log(1 - i\sqrt{2\lambda}\sigma)} \end{aligned}$$

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We recognize $\tau_c(z) = \sqrt{z}T(z)$ where $T(z) = \sum_n C_n z^n = \frac{1}{2z}[1 - \sqrt{1-4z}]$.

$$\begin{aligned} \lim_{N\to\infty} \frac{\log Z(\lambda, N)}{N} &= -f(\tau_c) = -\tau_c^2/2 - \log(1 - \sqrt{z}\tau_c)] \\ &= \frac{-1}{4z}[1 - \sqrt{1-4z} - 2z] - \log[\frac{1}{2}(1 + \sqrt{1-4z})]. \end{aligned}$$

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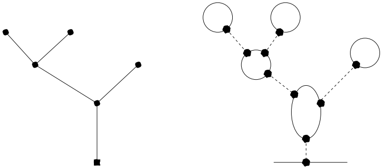

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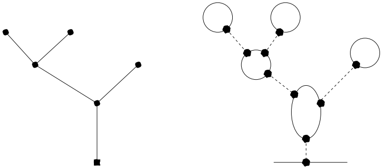

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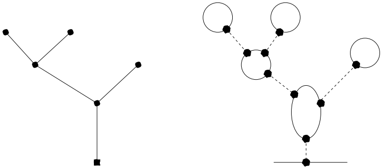

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- $(1/N)^p$-sub leading term correspond to add exactly p loops. Double scaling would include all graphs hence unstable at $z > 0$.

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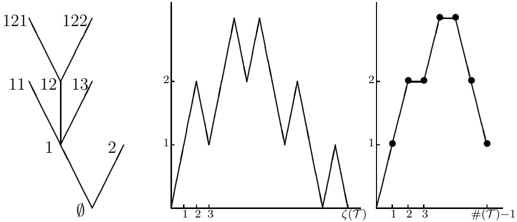

tree **t** contour function height function

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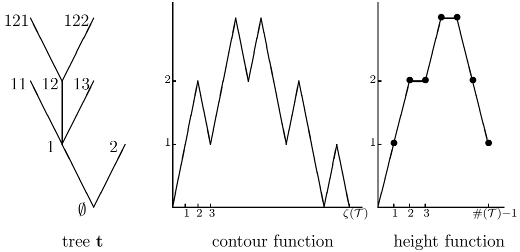
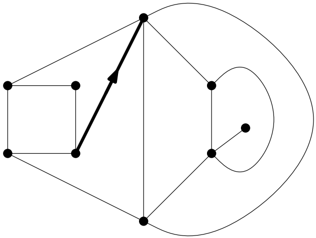


This Dyck map easily proves $d_{Hausdorff} = 2$; with further work $d_{spectral=4/3}$.

Planar Rooted Quadrangulations



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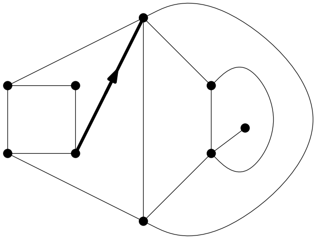

Planar maps, such as planar rooted quadrangulations, are natural combinatoric objects

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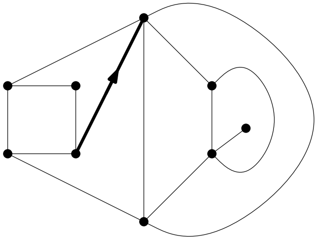

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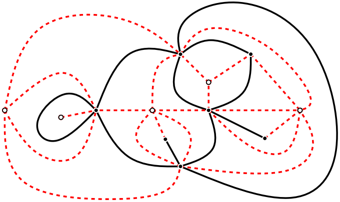

Counting Planar Rooted Quadrangulations

Q_n= number of rooted planar quadrangulations with n faces

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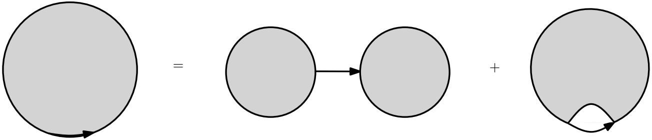

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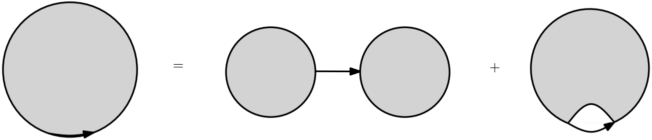

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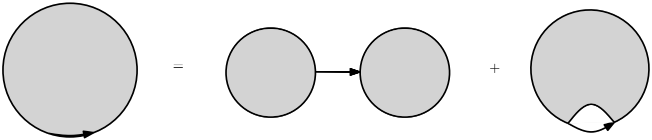

and solved it, getting:

$$Q_n = 3^n \frac{2}{n+2} \frac{1}{n+1} \binom{2n}{n} .$$

Planar Graphs and Matrix Models

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$$
2 - 2g = V - L + F = -V + F \quad , \quad \text{since} \quad L = 2V.
$$

hence planar graphs lead at $N \to \infty$ ('t Hooft, 1974).

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- Double scaling: $N \to \infty$ and $\lambda \to \lambda_c = 1/24$, $\kappa^{-1} = N^5/4(\lambda - \lambda_c)$ fixed

$$G_{2,double\ scaling}(\lambda) = \sum_h a_h \kappa^{2h}$$

includes all graphs, at $\lambda > 0$, hence unstable.

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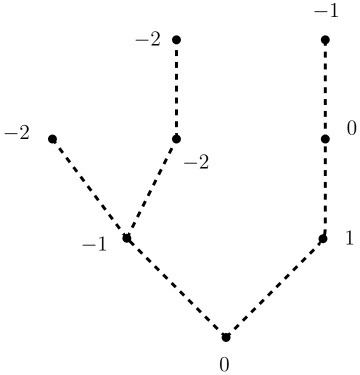
$$(n+2)Q_n = 2 \cdot 3^n C_n, \ .$$

The Schaeffer map identifies rooted, pointed planar quadrangulations with well-labeled, oriented rooted plane trees.

The Schaeffer Map



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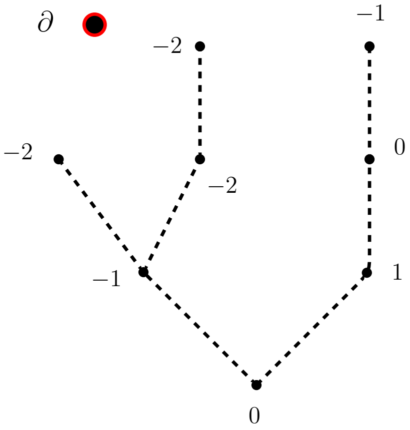

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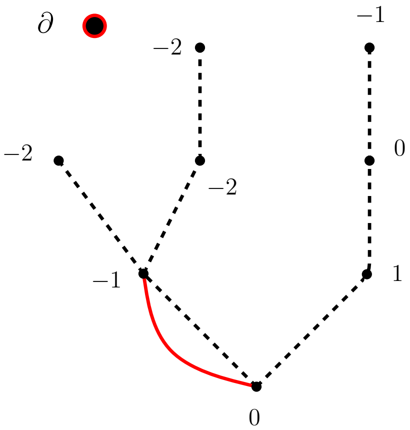

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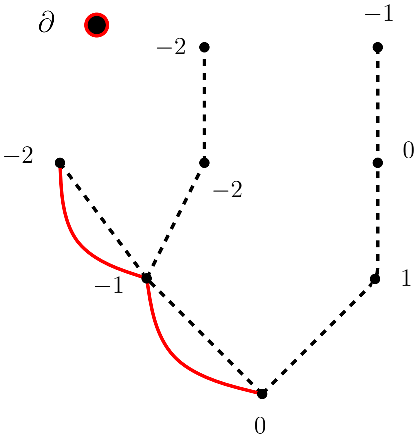

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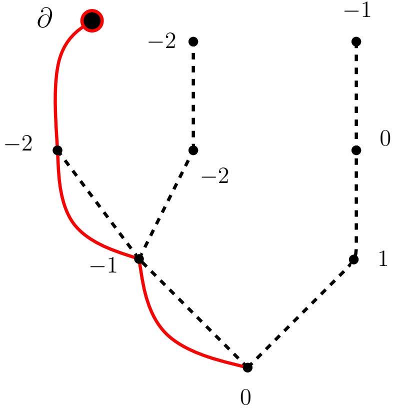

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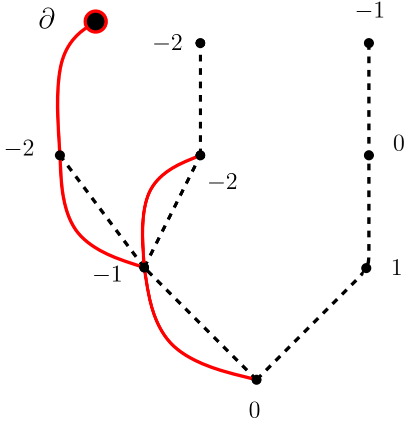

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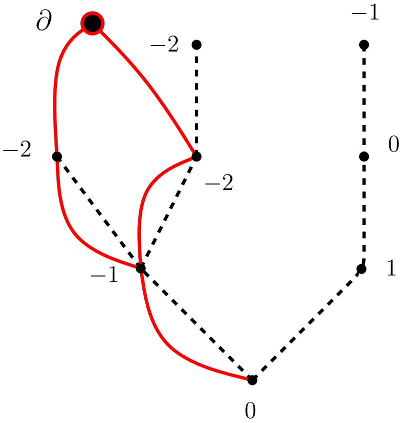

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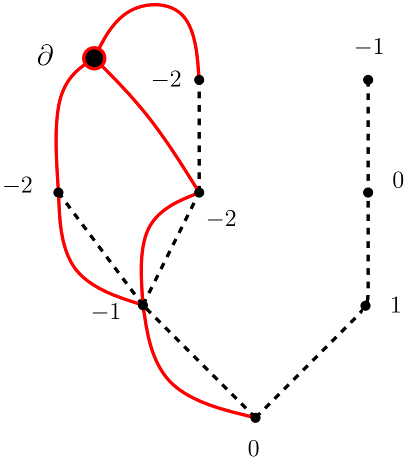

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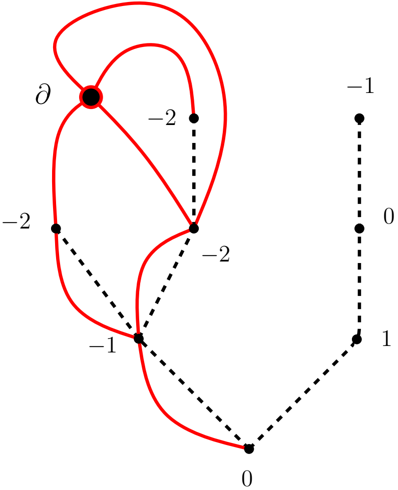

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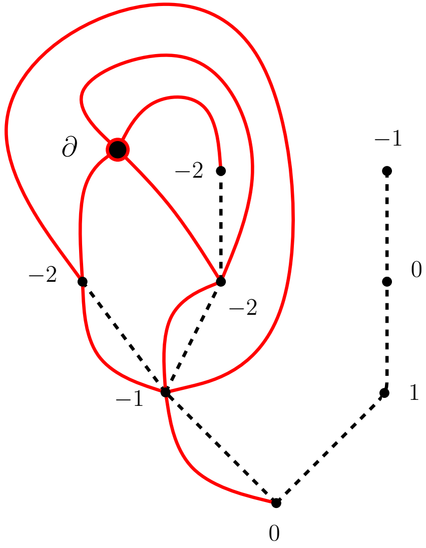

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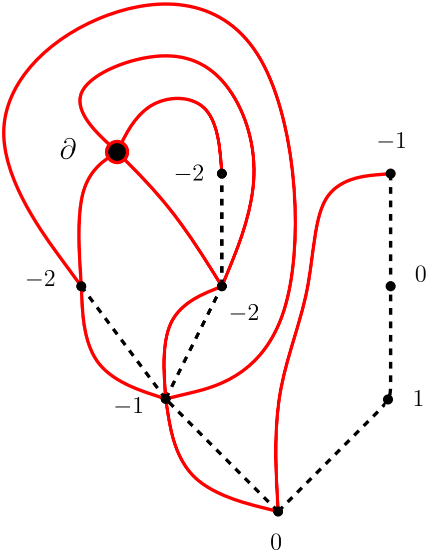

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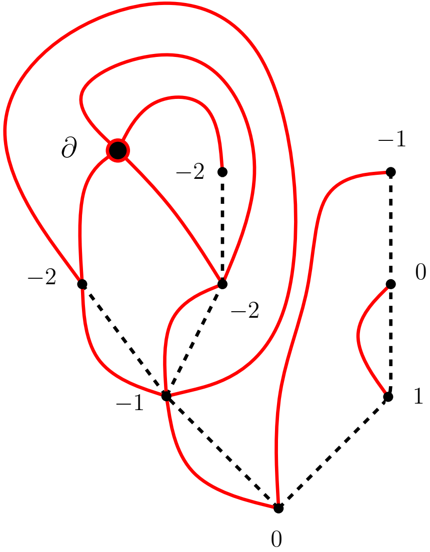
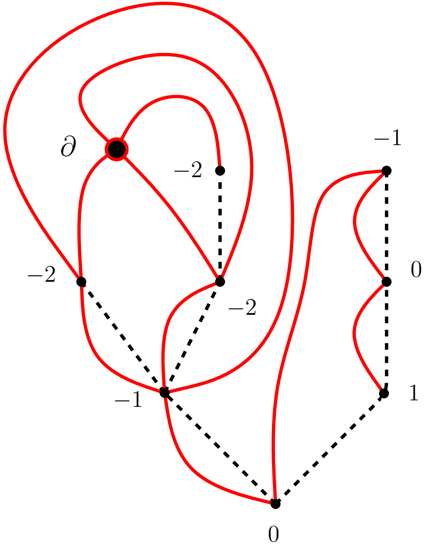


Large Quadrangulations



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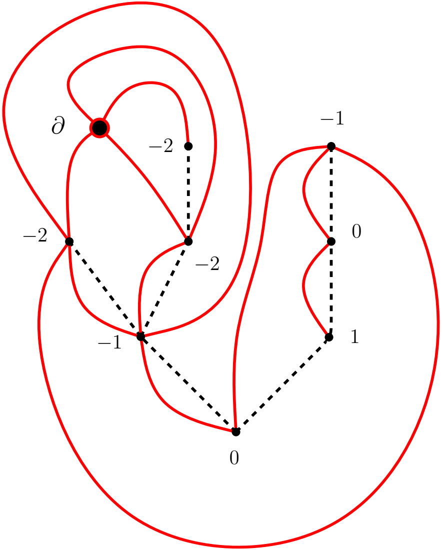

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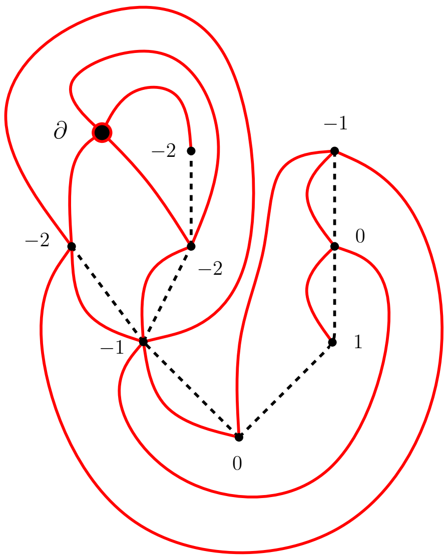

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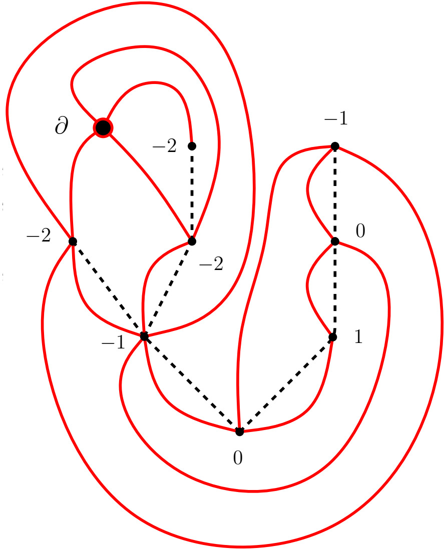

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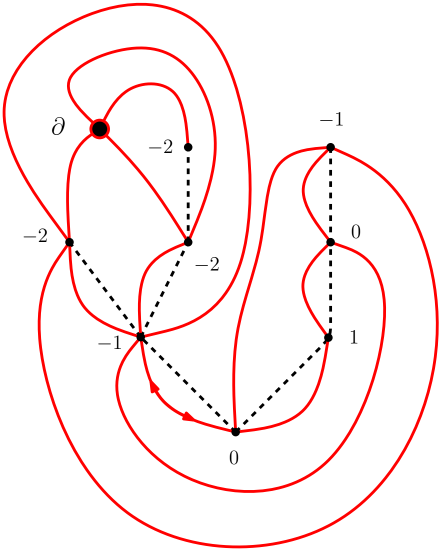

The two-Dimensional Brownian Sphere

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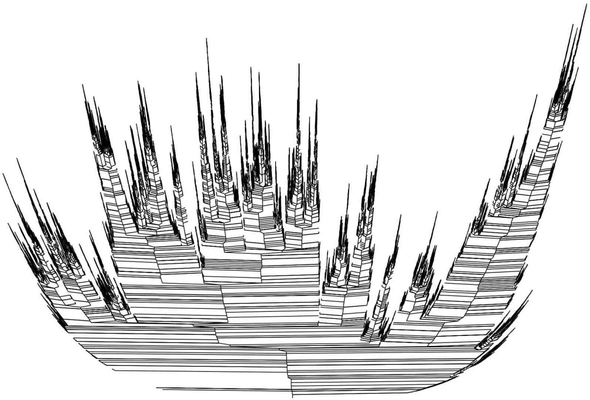
A Look at Large Random Quadrangulations



The Probabilist's View: The Brownian Snake, Head on

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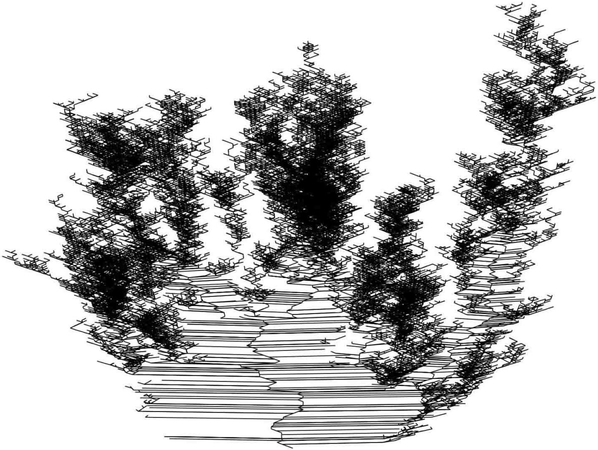

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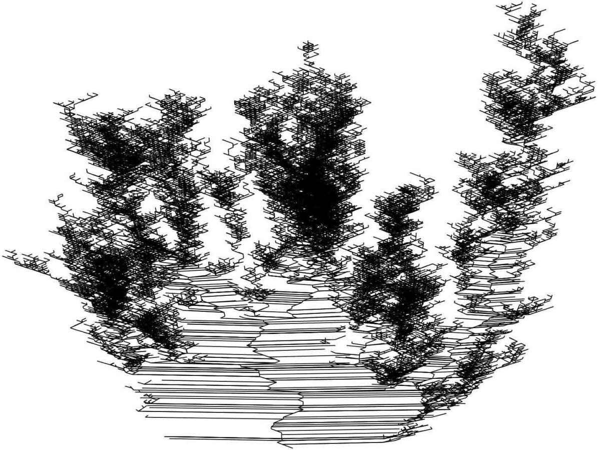

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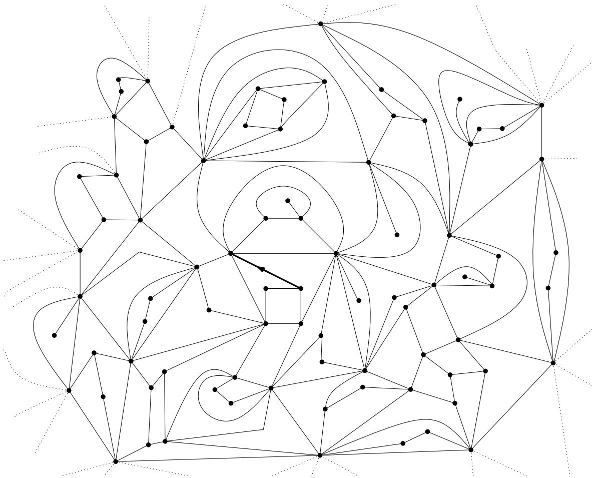

The Topological View

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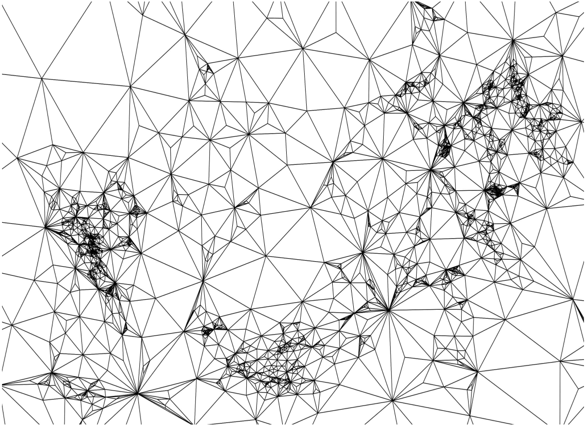

Uniformized Through Riemann Mapping Theorem

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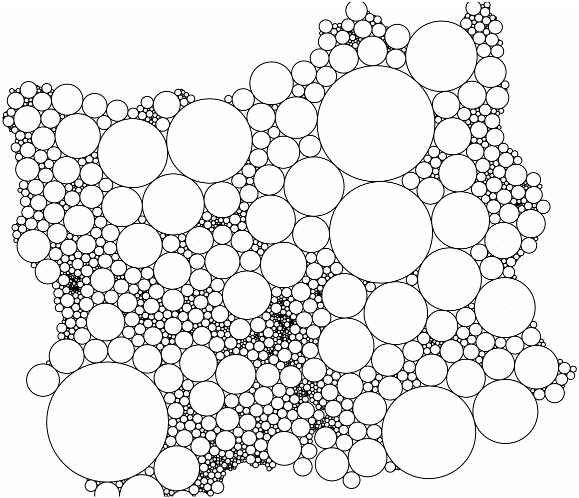

Using the Circle Packing Theorem (Courtesy: Krikun)

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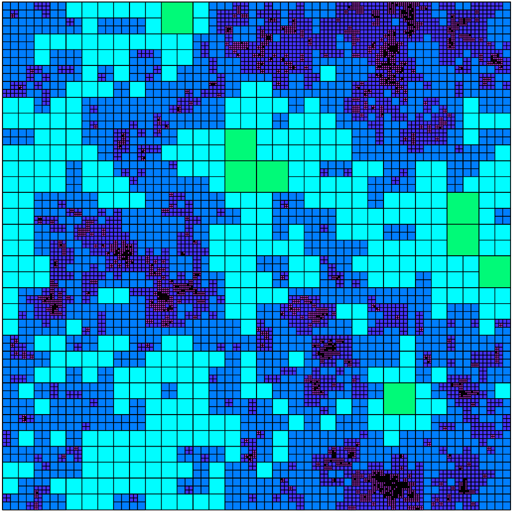

The Liouville Theory (Courtesy: Duplantier)

Physical Consequences: the KPZ relations

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Δ and x are scaling Euclidean dimensions coupled or not to quantum gravity plus matter with conformal central charge c (for pure gravity $c = 0, \gamma = \sqrt{8/3}$; for Ising, $c = 1/2, \gamma = \sqrt{3}$).

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- This Brownian sphere is best thought of as a CRT equipped with fluctuation fields, the Schaeffer labels, which add loops, hence space-time shortcuts.
- What about higher dimensions QG3, QG4... ?